

GALOIS COMODULES

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ABSTRACT. Galois comodules of a coring are studied. The conditions for a simple comodule to be a Galois comodule are found. A special class of Galois comodules termed principal comodules is introduced. These are defined as Galois comodules that are projective over their comodule endomorphism rings. A complete description of principal comodules in the case a background ring is a field is found. In particular it is shown that a (finitely generated and projective) right comodule of an A -coring \mathcal{C} is principal provided a lifting of the canonical map is a split epimorphism in the category of left \mathcal{C} -comodules. This description is then used to characterise principal extensions or non-commutative principal bundles. Specifically, it is proven that, over a field, any entwining structure consisting of an algebra A , a coseparable coalgebra C and a bijective entwining map ψ together with a group-like element in C give rise to a principal extension provided the lifted canonical map is surjective. Induction of Galois and principal comodules via morphisms of corings is described. A connection between the relative injectivity of a Galois comodule and the properties of the extension of endomorphism rings associated to this comodule is revealed.

1. INTRODUCTION

In an attempt to achieve a better conceptual understanding of a generalisation of a Hopf-Galois extension known as a coalgebra-Galois extension, the notion of a *Galois coring* has been introduced in [5], and recently investigated in [11], [24]. Following a classical route in the ring extension theory along which the properties of an extension are encoded in properties of a module (cf. [22]), it has been realised in [14] that the proper framework for studying Galois corings is provided by a certain class of its comodules, known as *Galois comodules*. The most important result about Galois comodules is the Galois comodule structure theorem formulated in [14, Theorem 3.2] (see Theorem 2.1 below) that incorporates the Galois Coring Theorem in [5, Theorem 5.6], from which in turn one of Schneider's theorems on the structure of Hopf-Galois extensions [21, Theorem 3.7] can be deduced.

The aim of the present paper is to study properties of Galois comodules, as a means for providing deeper conceptual understanding of Galois-type extensions, in particular those that are motivated by non-commutative geometry (where they appear as non-commutative principal bundles). In particular, Schneider's theorem [21, Theorem 3.7]

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is known as an *easy* (properly, *descent theory*) part of a full structure theorem [21, Theorem I] for Hopf-Galois extensions, recently beautifully extended to a class of coalgebra-Galois extensions in [20]. The difficult part of [21, Theorem I] involves showing that, when appropriate assumptions are made, the bijectivity of the canonical Galois map follows from its surjectivity. In Section 3 we show that also in the case of a simple Galois comodule of a coring, the surjectivity of the canonical map implies injectivity. In Section 4 we concentrate on Galois comodules which are projective over their endomorphism rings. We term such Galois comodules *principal comodules*. The interest in such comodules stems from non-commutative geometry, in particular from the theory of *strong connections* [17] in coalgebra-Galois extensions understood as non-commutative principal bundles [8]. A certain class of such extensions, identified and systematised in [7] as *principal extensions* has non-commutative vector bundles, understood as finitely generated projective modules (cf. [12]), as their associated fibre bundles. Principal comodules defined in the present paper seem to provide a suitable general framework for principal extensions. We derive a full characterisation of principal comodules in the case when the background ring is a field. This is related to the (split-)surjectivity of certain lifting of the canonical map (and hence again resembles the difficult part of Schneider's theorem). We then use this description to prove that, over a field, any entwining structure consisting of an algebra A , a coseparable coalgebra C and a bijective entwining map ψ together with a group-like element in C give rise to a principal extension provided the lifted canonical map is surjective. This can be understood as the entwining structure version of the difficult part of Schneider's Theorem I, and extends recent theorem of Schauenburg and Schneider [20, Theorem 2.5.7] formulated for a class of Doi-Koppinen entwinnings. It also means geometrically that in this case a freeness of the group action induces existence of a (strong) connection on the corresponding principal fibre bundle.

Second class of problems addressed in this paper involves questions, what properties of comodules are preserved by morphisms of corings. More precisely, any morphism of corings \mathcal{C} to \mathcal{D} induces a \mathcal{D} -comodule from a \mathcal{C} -comodule. If a \mathcal{C} -comodule is a Galois comodule, is the induced \mathcal{D} -comodule also a Galois \mathcal{D} -comodule? Thus in Section 5 we determine which morphisms of corings induce principal comodules from principal comodules. The importance of this induction procedure of principal comodules, and, in particular, principal extensions, in non-commutative geometry has been confirmed in recent work [1] in which it has been shown that the non-commutative 4-sphere and the corresponding instanton bundle constructed in [2]

arise from a principal extension of this type. Furthermore, the functor inducing \mathcal{D} -comodules from \mathcal{C} -comodules features prominently in the Kontsevich-Rosenberg approach to non-commutative algebraic geometry [18], where it is understood as a pull-back of quasi-coherent sheaves over non-commutative stacks, while principal comodules are examples of covers of non-commutative spaces.

The third class of problems discussed in this paper is concerned with duality properties of Galois comodules and with the relative injectivity of Galois comodules. In Section 6, we study modules (of the endomorphism ring of a Galois comodule) associated to Galois comodules by applying the Hom-functor. The motivation of this construction comes from non-commutative geometry, where modules of this kind are understood as fibre bundles associated to non-commutative (coalgebra) principal bundles (cf. [4]). We reveal a remarkable duality with respect to the change of arguments in the Hom-bifunctor. This can again be understood in geometric terms as the (generalisation of the) identification of sections of a vector bundle with tensorial zero-forms (functions of type ρ). We describe a sufficient condition on a Galois comodule M of an A -coring \mathcal{C} that makes any of these associated modules a finitely generated projective module (provided a “fibre” comodule is a finitely generated projective A -module). Finally in Section 7, we connect the relative injectivity of a Galois comodule with the properties of the inclusion of the comodule endomorphism rings into the module endomorphism ring. This connection then leads to a criterion for faithful flatness of a Galois comodule that generalises the criterion introduced for Hopf-Galois extensions in [13, Theorem 2.11] and for coalgebra-Galois extensions in [4, Proposition 4.4]. We also show that if the extension of endomorphism rings of a principal comodule is a split extension, then any module, associated in the way discussed in Section 6, is a finitely generated projective module over the endomorphism ring of the principal comodule (provided a “fibre” comodule is a finitely generated projective A -module).

2. REVIEW OF CORINGS AND THE GALOIS COMODULE STRUCTURE THEOREM

We work over a commutative ring k with a unit. All algebras are over k , associative and with a unit. All coalgebras are over k , coassociative and with a counit. In a coalgebra C , the coproduct is denoted by Δ_C and the counit by ε_C . The identity morphism for an object V is also denoted by V . For a ring (k -algebra) R , the category of right R -modules and right R -linear maps is denoted by \mathbf{M}_R . Symmetric notation is used for left modules. As is customary, we often write M_R to indicate that M is a

right R -module, etc. When needed, a right action of R on M_R is denoted by ϱ_M and the left action of R on ${}_R N$ is denoted by ${}_N \varrho$. On elements, the actions are denoted by juxtaposition. The dual module of M_R is denoted by M^* , while the dual of ${}_R N$ is denoted by ${}^* N$. The product in the endomorphism ring of a right module (comodule) is given by composition of maps, while the product in the endomorphism ring of a left module (comodule) is given by opposite composition (we always write argument to the right of a function).

Let A be an algebra. A coproduct in an A -coring \mathcal{C} is denoted by $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$, and the counit is denoted by $\varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$. To indicate the action of $\Delta_{\mathcal{C}}$ we use the Sweedler sigma notation, i.e., for all $c \in \mathcal{C}$,

$$\Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes c_{(2)}, \quad (\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}}(c) = (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$

etc. Calligraphic capital letters always denote corings. The category of right \mathcal{C} -comodules and right \mathcal{C} -colinear maps is denoted by $\mathbf{M}^{\mathcal{C}}$. Recall that $\mathbf{M}^{\mathcal{C}}$ is built upon the category of right A -modules, in the sense that there is a forgetful functor $\mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$. In particular, any right \mathcal{C} -comodule is also a right A -module, and any right \mathcal{C} -comodule map is right A -linear. For a right \mathcal{C} -comodule M , $\varrho^M : M \rightarrow M \otimes_A \mathcal{C}$ denotes a coaction, and $\text{Hom}^{-\mathcal{C}}(M, N)$ is the k -module of \mathcal{C} -colinear maps $M \rightarrow N$. On elements ϱ^M is denoted by the Sweedler notation $\varrho^M(m) = \sum m_{(0)} \otimes m_{(1)}$. Symmetric notation is used for left \mathcal{C} -comodules. In particular, the coaction of a left \mathcal{C} -comodule N is denoted by ${}_N \varrho$, and, on elements, by ${}_N \varrho(n) = \sum n_{(-1)} \otimes n_{(0)} \in \mathcal{C} \otimes_A N$. Of course, coalgebras are examples of corings, hence the same rules of notation for comodules over a coalgebra as those for comodules over a coring apply. A detailed account of the theory of corings and comodules can be found in [9].

Given a right \mathcal{C} -comodule M and a left \mathcal{C} -comodule N one defines a *cotensor product* $M \square_{\mathcal{C}} N$ by the following exact sequence of k -modules:

$$0 \longrightarrow M \square_{\mathcal{C}} N \longrightarrow M \otimes_A N \xrightarrow{\omega_{M,N}} M \otimes_A \mathcal{C} \otimes_A N,$$

where $\omega_{M,N} = \varrho^M \otimes_A N - M \otimes_A {}^N \varrho$ (i.e., $M \square_{\mathcal{C}} N$ is an equaliser of $\varrho^M \otimes_A N$ and $M \otimes_A {}^N \varrho$, where ϱ^M and ${}_N \varrho$ are coactions). $M \square_{\mathcal{C}} N$ is a left S -module of the endomorphism ring $S = \text{End}^{-\mathcal{C}}(M)$, via $s(\sum_i m_i \otimes n_i) = \sum_i s(m_i) \otimes n_i$, for all $s \in S$, $\sum_i m_i \otimes n_i \in M \square_{\mathcal{C}} N$.

Let \mathcal{C} be an A -coring and let \mathcal{D} be a B -coring. A morphism of corings is a pair (α, γ) , where $\alpha : A \rightarrow B$ is an algebra map and $\gamma : \mathcal{C} \rightarrow \mathcal{D}$ is an (A, A) -bimodule map such that

$$\chi \circ (\gamma \otimes_A \gamma) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ \gamma, \quad \varepsilon_{\mathcal{D}} \circ \gamma = \alpha \circ \varepsilon_{\mathcal{C}},$$

where $\chi : \mathcal{D} \otimes_A \mathcal{D} \rightarrow \mathcal{D} \otimes_B \mathcal{D}$ is the canonical morphism of (A, A) -bimodules induced by α . The (A, A) -bimodule structure of \mathcal{D} is induced from the (B, B) -bimodule structure via the map α (i.e., $ada' = \alpha(a)d\alpha(a')$, for all $a, a' \in A$ and $d \in \mathcal{D}$). Such a morphism of corings is explicitly denoted by $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$. In this case any right \mathcal{C} -comodule M gives rise to a right \mathcal{D} -comodule $M \otimes_A B$ with the coaction

$$\varrho^{M \otimes_A B} : M \otimes_A B \rightarrow M \otimes_A B \otimes_B \mathcal{D} \simeq M \otimes_A \mathcal{D}, \quad m \otimes b \mapsto \sum m_{(0)} \otimes \gamma(m_{(1)})b.$$

For any right \mathcal{C} -comodule M and a right \mathcal{D} -comodule N , $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, N)$ is a right S -module of the endomorphism ring $S = \text{End}^{-\mathcal{C}}(M)$ via $fs(m \otimes b) = f(s(m) \otimes b)$, for all $s \in S$, $f \in \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N)$, $m \in M$ and $b \in B$.

Symmetrically, any left \mathcal{C} -comodule N gives rise to a left \mathcal{D} -comodule $B \otimes_A N$. In particular $B \otimes_A \mathcal{C}$ is a left \mathcal{D} -comodule with the coaction

$${}^{B \otimes_A \mathcal{C}}\varrho : B \otimes_A \mathcal{C} \rightarrow \mathcal{D} \otimes_B B \otimes_A \mathcal{C} \simeq \mathcal{D} \otimes_A \mathcal{C}, \quad b \otimes c \mapsto \sum b\gamma(c_{(1)}) \otimes c_{(2)}.$$

Thus for any morphism of corings $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$, there is an associated pair of functors

$$-\otimes_A B : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}^{\mathcal{D}}, \quad -\square_{\mathcal{D}}(B \otimes_A \mathcal{C}) : \mathbf{M}^{\mathcal{D}} \rightarrow \mathbf{M}_A.$$

If \mathcal{C} is flat as a left A -module, one shows that $-\square_{\mathcal{D}}(B \otimes_A \mathcal{C}) : \mathbf{M}^{\mathcal{D}} \rightarrow \mathbf{M}^{\mathcal{C}}$, and it is a right adjoint of $-\otimes_A B$. We refer to [9, Section 24] for more details about morphisms of corings and associated functors. By an *A-coring morphism*, a morphism of corings $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : A)$ is meant in which α is the identity map (so that only γ needs to be specified).

Given right \mathcal{C} -comodules M and N , the k -module $\text{Hom}^{-\mathcal{C}}(M, N)$ is a right module of the endomorphism ring $S = \text{End}^{-\mathcal{C}}(M)$ with the standard action $fs = f \circ s$, for all $f \in \text{Hom}^{-\mathcal{C}}(M, N)$, $s \in S$. This defines a functor $\text{Hom}^{-\mathcal{C}}(M, -) : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_S$. The functor $\text{Hom}^{-\mathcal{C}}(M, -)$ has the left adjoint $-\otimes_S M : \mathbf{M}_S \rightarrow \mathbf{M}^{\mathcal{C}}$, where, for any $X \in \mathbf{M}_S$, $X \otimes_S M$ is a right \mathcal{C} -comodule with the coaction $X \otimes_S \varrho^M$. The counit of the adjunction is given by the evaluation map

$$\varphi_N : \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M \rightarrow N, \quad f \otimes m \mapsto f(m),$$

while the unit is $\nu_X : X \rightarrow \text{Hom}^{-\mathcal{C}}(M, X \otimes_S M)$, $x \mapsto [m \mapsto x \otimes m]$ (cf. [9, 18.21]).

This paper is concerned with a special class of comodules introduced in [14] and known as *Galois comodules*. The properties of these comodules reflect properties of the above pair of adjoint functors. Let \mathcal{C} be an A -coring, M be a right \mathcal{C} -comodule and let $S = \text{End}^{-\mathcal{C}}(M)$. View \mathcal{C} as a right \mathcal{C} -comodule with the regular coaction $\Delta_{\mathcal{C}}$.

M is called a *Galois (right) comodule* if M is a finitely generated and projective right A -module, and the evaluation map

$$\varphi_{\mathcal{C}} : \text{Hom}^{-\mathcal{C}}(M, \mathcal{C}) \otimes_S M \rightarrow \mathcal{C}, \quad f \otimes m \mapsto f(m),$$

is an isomorphism of right \mathcal{C} -comodules.

An equivalent definition of Galois comodules is obtained by first noting that M is an (S, A) -bimodule and $\text{Hom}^{-\mathcal{C}}(M, \mathcal{C}) \simeq M^* = \text{Hom}_{-A}(M, A)$ as (A, S) -bimodules. If M_A is finitely generated projective, then $M^* \otimes_S M$ is an A -coring with the coproduct $\Delta_{M^* \otimes_S M}(\xi \otimes m) = \sum_i \xi \otimes e^i \otimes \xi^i \otimes m$, where $\{e^i \in M, \xi^i \in M^*\}$ is a dual basis of M_A , and with the counit $\varepsilon_{M^* \otimes_S M}(\xi \otimes m) = \xi(m)$ (cf. [14]). The map $\varphi_{\mathcal{C}}$ reduces to the *canonical* A -coring morphism

$$\text{can}_M : M^* \otimes_S M \rightarrow \mathcal{C}, \quad \xi \otimes m \mapsto \sum \xi(m_{(0)})m_{(1)}.$$

M (with M_A finitely generated projective) is a Galois comodule if and only if the canonical map can_M is an isomorphism of corings.

The case in which A is a Galois \mathcal{C} -comodule is of fundamental importance. In this case the coaction $\varrho^A : A \rightarrow A \otimes_A \mathcal{C} \simeq \mathcal{C}$ is fully determined by a group-like element $g = \varrho^A(1) \in \mathcal{C}$. The endomorphism ring $S = \text{End}^{-\mathcal{C}}(A)$ coincides with the subalgebra of g -coinvariants in A , i.e., $S = \{s \in A \mid sg = gs\}$. Obviously, A is a finitely generated projective right A -module, $A^* \simeq A$, and $A \otimes_S A$ is the *Sweedler A -coring*, with coproduct $a \otimes a' \mapsto a \otimes 1 \otimes 1 \otimes a'$ and counit $a \otimes a' \mapsto aa'$. The canonical map comes out as

$$\text{can}_A : A \otimes_S A \rightarrow \mathcal{C}, \quad a \otimes a' \mapsto aga'.$$

Thus A is a Galois comodule if and only if \mathcal{C} is a *Galois coring* with respect to g , a notion introduced in [5].

Main properties of Galois comodules are contained in the Galois comodule structure theorem, which, in part, was first formulated in [14, Theorem 3.2].

Theorem 2.1. (The Galois comodule structure theorem) *Let \mathcal{C} be an A -coring and M be a right \mathcal{C} -comodule that is finitely generated and projective as a right A -module. Set $S = \text{End}^{-\mathcal{C}}(M)$.*

(1) *The following are equivalent:*

- (a) *M is a Galois comodule that is flat as a left S -module.*
- (b) *\mathcal{C} is a flat left A -module and M is a generator in $\mathbf{M}^{\mathcal{C}}$.*
- (c) *\mathcal{C} is a flat left A -module and, for any $N \in \mathbf{M}^{\mathcal{C}}$, the counit of adjunction $\varphi_N : \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M \rightarrow N$ is an isomorphism of right \mathcal{C} -comodules.*

(2) *The following are equivalent:*

- (a) *M is a Galois comodule that is faithfully flat as a left S -module.*
- (b) *\mathcal{C} is a flat left A -module and M is a projective generator in $\mathbf{M}^{\mathcal{C}}$.*
- (c) *\mathcal{C} is a flat left A -module and $\mathrm{Hom}^{-\mathcal{C}}(M, -) : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_S$ is an equivalence with the inverse $- \otimes_S M : \mathbf{M}_S \rightarrow \mathbf{M}^{\mathcal{C}}$.*

For the proof of this theorem we refer to [9, 18.27] and only point out that the equivalence (b) \Leftrightarrow (c) in (1) is a consequence of the description of generators in $\mathbf{M}^{\mathcal{C}}$ as static comodules in [9, 18.23].

3. SIMPLE GALOIS COMODULES

The aim of this section is to prove that, for a simple \mathcal{C} -comodule M , to show that M is a Galois comodule suffices it to check whether the map φ_M is surjective. Recall that an object M in an Abelian category is a *simple object* provided every monomorphism $N \rightarrow M$ is either 0 or an isomorphism. The following characterisation of simple comodules extends a theorem of Takeuchi reported in [19].

Theorem 3.1. *Let \mathcal{C} be an A -coring that is flat as a left A -module. Let M be a right \mathcal{C} -comodule and let $S = \mathrm{End}^{-\mathcal{C}}(M)$ be its endomorphism ring. Then the following are equivalent:*

- (1) *M is a simple comodule, i.e., a simple object in $\mathbf{M}^{\mathcal{C}}$.*
- (2) *S is a division ring and for any right \mathcal{C} -comodule N , the evaluation map*

$$\varphi_N : \mathrm{Hom}^{-\mathcal{C}}(M, N) \otimes_S M \rightarrow N, \quad f \otimes m \mapsto f(m),$$

is a monomorphism in $\mathbf{M}^{\mathcal{C}}$.

Proof. Since \mathcal{C} is flat as a left A -module, the category $\mathbf{M}^{\mathcal{C}}$ is a Grothendieck category (cf. [9, 18.14]), hence, in particular, it is Abelian.

(1) \Rightarrow (2) If M is a simple comodule, then S is a division ring by the Schur lemma. Thus we need only to show that, for any $N \in \mathbf{M}^{\mathcal{C}}$, the map φ_N has a trivial kernel. Note that any element of $\mathrm{Hom}^{-\mathcal{C}}(M, N) \otimes_S M$ (and hence also of the kernel of φ_N) can be written as a finite sum $\sum_i f_i \otimes m_i$ with the $f_i \in \mathrm{Hom}^{-\mathcal{C}}(M, N)$ and $m_i \in M$. Since S is a division ring we can always choose the f_i in such a way that they form a free set in the right S -module $\mathrm{Hom}^{-\mathcal{C}}(M, N)$, and we always choose the f_i in this way. Suppose that a simple tensor $f \otimes m$ is in the kernel of φ_N , i.e., that $f(m) = 0$. This means that $m \in \ker f$. On the other hand M is a simple object so that the inclusion monomorphism $0 \rightarrow \ker f \rightarrow M$ is either 0 or an isomorphism. In the first

case the kernel of f is trivial, hence $m = 0$, and therefore $f \otimes m = 0$. In the other case every $m \in M$ is in the kernel of M , hence $f = 0$ and $m \otimes f = 0$. Thus the kernel of φ_N does not contain any non-trivial simple tensors.

Now assume inductively that any non-trivial element consisting of less than n simple tensors cannot be in the kernel of φ_N , i.e., that $\sum_{i=1}^{n-1} f_i(m_i) = 0$ implies that $m_1 = m_2 = \dots = m_{n-1} = 0$ (as explained, we choose the f_i in such a way that they form a free set). Suppose to the contrary that there exist non-zero $m_n \in M$ and $f_n \in \text{Hom}^{-\mathcal{C}}(M, N)$ such that

$$f_1(m_1) + f_2(m_2) + \dots + f_n(m_n) = 0,$$

and $\{f_1, f_2, \dots, f_n\}$ is a free set in the right S -module $\text{Hom}^{-\mathcal{C}}(M, N)$. This implies that

$$f_n(M) \cap (\oplus_{k=1}^{n-1} f_k(M)) \neq 0.$$

Next observe that $f_n(M) \simeq f_n S \otimes_S M \simeq M$ via the isomorphism $f_n s \otimes m \mapsto sm$, well-defined because S is a division ring. Since M is a simple object, so is $f_n(M) \simeq M$, thus the above intersection property implies that

$$f_n(M) \subset \oplus_{k=1}^{n-1} f_k(M). \quad (*)$$

Note that $f_k(M) = f_k S \otimes_S M$, and since every $f_k S$ is a finitely generated free S -module, there is the following chain of isomorphisms

$$\text{Hom}^{-\mathcal{C}}(M, f_k(M)) \simeq f_k S \otimes_S \text{Hom}^{-\mathcal{C}}(M, M) = f_k S \otimes_S S \simeq f_k S.$$

Applying $\text{Hom}^{-\mathcal{C}}(M, -)$ to the inclusion $(*)$ we thus obtain

$$f_n S \subset \text{Hom}^{-\mathcal{C}}(M, \oplus_{k=1}^{n-1} f_k(M)) = \oplus_{k=1}^{n-1} \text{Hom}^{-\mathcal{C}}(M, f_k(M)) \simeq \oplus_{k=1}^{n-1} f_k S.$$

This, however, contradicts the assumption that the set $\{f_1, f_2, \dots, f_n\}$ is S -free. Hence $f_1(m_1) + f_2(m_2) + \dots + f_n(m_n) = 0$ implies that $m_1 = m_2 = \dots = m_n = 0$ and $\ker \varphi_N = 0$ by induction.

(2) \Rightarrow (1) Let $f : J \rightarrow M$ be a monomorphism in $\mathbf{M}^{\mathcal{C}}$ and let $N = \text{coker } f$. Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} 0 & \longrightarrow & S \otimes_S M & \xrightarrow{\simeq} & M & \longrightarrow & 0 \\ & & \pi \otimes_S M \downarrow & & \downarrow p & & \\ 0 & \longrightarrow & \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M & \xrightarrow{\varphi_N} & N & \longrightarrow & 0, \\ & & & & \downarrow & & \\ & & & & 0, & & \end{array}$$

where p is the canonical epimorphism and the map $\pi : S \rightarrow \text{Hom}^{-\mathcal{C}}(M, N)$ is given by $\pi(s)(m) = p(s(m))$. It follows that φ_N is an isomorphism, and therefore, $\pi \otimes_S M$ is an epimorphism. Since S is a division ring, M is a faithfully flat left S -module, hence also π is an epimorphism. Furthermore, since π is a right S -linear map and S is a division ring, $\ker \pi = 0$ or $\ker \pi = S$. If $\ker \pi = 0$, then π is an isomorphism, and so is p , thus $\text{coker } f \simeq M$ and therefore f is a zero map. If $\ker \pi = S$, then π is the zero map, so $\text{Hom}^{-\mathcal{C}}(M, N) = 0$, i.e., $\text{coker } f = 0$ and f is an isomorphism. Thus M is a simple object. \square

Theorem 3.1 leads to the following description of simple Galois comodules.

Corollary 3.2. *Let \mathcal{C} be an A -coring that is flat as a left A -module. Then:*

- (1) *Every Galois comodule whose endomorphism ring is a division ring is a simple comodule.*
- (2) *Let M be a simple right \mathcal{C} -comodule that is finitely generated and projective as a right A -module, and let $S = \text{End}^{-\mathcal{C}}(M)$. Then the following are equivalent:*
 - (a) *M is a Galois comodule.*
 - (b) *For all $N \in \mathbf{M}^{\mathcal{C}}$, the evaluation map $\varphi_N : \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M \rightarrow N$ is surjective.*
 - (c) *The evaluation map $\varphi_{\mathcal{C}} : \text{Hom}^{-\mathcal{C}}(M, \mathcal{C}) \otimes_S M \rightarrow \mathcal{C}$ is surjective.*
 - (d) *The canonical map $\text{can}_M : M^* \otimes_S M \rightarrow \mathcal{C}$, $\xi \otimes m \mapsto \sum \xi(m_{(0)})m_{(1)}$ is an epimorphism of A -corings.*

Proof. (1) If M is a Galois comodule with endomorphism ring S that is a division ring, then M is a flat left S -module, hence φ_N is an isomorphism for any $N \in \mathbf{M}^{\mathcal{C}}$ by Theorem 2.1. Thus M is a simple object in $\mathbf{M}^{\mathcal{C}}$ by Theorem 3.1.

(2) Since M is a simple comodule, S is a division ring, and every \mathcal{C} -comodule is a flat left S -module. Hence the implications (a) \Rightarrow (b) \Rightarrow (c) are obvious. In view of Theorem 3.1, the evaluation map $\varphi_{\mathcal{C}}$ is injective, so condition (c) implies condition (a). Finally the equivalence (c) \Leftrightarrow (d) follows from the hom-tensor isomorphism $\text{Hom}^{-\mathcal{C}}(M, \mathcal{C}) \simeq \text{Hom}^{-\mathcal{C}}(M, A \otimes_A \mathcal{C}) \simeq \text{Hom}_{-A}(M, A)$. \square

Note in passing that, by extracting the key features of Theorem 3.1 and using a metatheorem of Abelian categories (cf. [15, Chapter 4]), one can obtain a characterisation of simple objects in general Abelian categories. The key features are that the functor $- \otimes_S M$ is the left adjoint of the functor $\text{Hom}^{-\mathcal{C}}(M, -)$ and that

the map φ_N is a counit of this adjunction. Furthermore, the statement of the theorem is a compound diagrammatic statement. Finally, the facts that A is a (trivial) A -coring, and the category of right A -comodules is isomorphic to the category of right A -modules assure that Theorem 3.1 holds for any category of modules. Thus a metatheorem of Abelian categories together with the Mitchell Embedding Theorem (cf. [15, Chapter 7]) lead to the following characterisation of simple objects. In any Abelian category \mathbf{C} , an object M such that $\text{Mor}_{\mathbf{C}}(M, -)$ has the left adjoint is simple if and only if its endomorphism ring is a division ring and, for any object, the counit of the adjunction is a monomorphism (compare characterisation of adjoints in [16, Chapitre V]). This is probably well-known to category theorists (although we were not able to find a reference).

4. PRINCIPAL COMODULES

In this section we introduce and study the following class of Galois comodules.

Definition 4.1. A Galois right \mathcal{C} -comodule M is said to be a *principal comodule* provided it is a projective left module of its endomorphism ring $S = \text{End}^{-\mathcal{C}}(M)$.

The prime interest in studying principal comodules stems from non-commutative geometry. One can argue that a principal comodule is as close an object as abstractly possible to the notion of a *principal extension* introduced recently in [7]. The latter is an example of a principal comodule of a coring associated to an *entwining structure*.

Recall from [8] that an entwining structure $(A, C)_{\psi}$ consists of of a k -algebra A , a k -coalgebra C and an k -module map $\psi : C \otimes_k A \rightarrow A \otimes_k C$ rendering commutative the following *bow-tie diagram*

$$\begin{array}{ccccc}
 & C \otimes_k A \otimes_k A & & C \otimes_k C \otimes_k A & \\
 \psi \otimes_k A \swarrow & & C \otimes_k \mu \searrow & \Delta_{C \otimes_k A} \nearrow & \\
 & C \otimes_k A & & & \\
 C \otimes_k \iota \nearrow & & \varepsilon_{C \otimes_k A} \searrow & & \\
 C & & A & & C \otimes_k A \otimes_k C \\
 \iota \otimes_k C \searrow & & A \otimes_k \varepsilon_C \nearrow & & \\
 & A \otimes_k C & & & \\
 A \otimes_k \psi \swarrow & & \mu \otimes_k C \nearrow & A \otimes_k \Delta_C \searrow & \\
 & A \otimes_k A \otimes_k C & & A \otimes_k C \otimes_k C & \\
 & & \psi \otimes_k C \nearrow & &
 \end{array} ,$$

where μ is the product in A and $\iota : k \rightarrow A$ is the unit map. The map ψ is known as an *entwining map*, and C and A are said to be *entwined* by ψ . As explained in [5], given an entwining structure $(A, C)_\psi$, $\mathcal{C} = A \otimes_k C$ is an A -coring with A -multiplications $a(a' \otimes c)a'' = aa'\psi(c \otimes a'')$, coproduct $\Delta_{\mathcal{C}} : A \otimes_k C \rightarrow A \otimes_k C \otimes_A A \otimes_k C \simeq A \otimes_k C \otimes_k C$, $a \otimes c \mapsto a \otimes \Delta_C(c)$, and counit $\varepsilon_{\mathcal{C}}(a \otimes c) = a\varepsilon_C(c)$. For more information about entwining structures and their connection with Hopf-type modules we refer to [10].

Example 4.2. Let k be a field, C be a coalgebra and A an algebra and a right C -comodule via $\varrho^A : A \rightarrow A \otimes_k C$. Let $S = A^{coC} := \{s \in A \mid \varrho^A(sa) = s\varrho^A(a), \forall a \in A\}$, denote the subalgebra of C -coinvariants of A . The inclusion of algebras $S \subseteq A$ is called a *principal C -extension* iff

- (1) $\text{can} : A \otimes_S A \rightarrow A \otimes_k C$, $a \otimes a' \mapsto a\varrho^A(a')$ is bijective (the Galois condition);
- (2) A is C -equivariantly projective as a left S -module, i.e., there exists a left S -module, right C -comodule section of the product $S \otimes_k A \rightarrow A$ (existence of a strong connection);
- (3) $\psi : C \otimes_k A \rightarrow A \otimes_k C$, $c \otimes a \mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a)$ is bijective (invertibility of the canonical entwining);
- (4) there is a group-like element $e \in C$ such that $\varrho^A(a) = \psi(e \otimes a)$, $\forall a \in A$ (co-augmentation).

If $S \subseteq A$ is a principal C -extension, then A is a principal comodule of the coring $\mathcal{C} = A \otimes_k C \simeq A \otimes_S A$.

Proof. By [6, Theorem 2.7], the map ψ entwines A with C , so $\mathcal{C} = A \otimes_k C$ is an A -coring of a type described above. Furthermore, A is a right \mathcal{C} -comodule with the coaction $\varrho^A : A \rightarrow A \otimes_k C \simeq A \otimes_A A \otimes_k C = A \otimes_A \mathcal{C}$. This means that $g = \varrho^A(1) = 1 \otimes e$ is a group-like element in \mathcal{C} , and $\varrho^A(a) = ga$ for all $a \in A$. By the obvious identification $\text{End}_{-A}(A) \simeq A$ we obtain

$$\text{End}^{-\mathcal{C}}(A) = \{f \in \text{End}_{-A}(A) \mid \forall a \in A, gf(a) = f(a)g\} = \{s \in A \mid sg = gs\} = S.$$

The Galois condition implies that \mathcal{C} is a Galois coring, hence $\mathcal{C} \simeq A \otimes_S A \simeq A^* \otimes_S A$, i.e., A is a Galois comodule. Finally, using the above identification of $\text{End}^{-\mathcal{C}}(A)$, the evaluation map $\text{End}^{-\mathcal{C}}(A) \otimes_k A \rightarrow A$ becomes simply the product map $S \otimes_k A \rightarrow A$, $s \otimes a \mapsto sa$, and hence the existence of a strong connection implies that A is a principal comodule. \square

A principal C -extension can be seen as a non-commutative version of a principal bundle. Since k is a field (this is the main case of interest from the point of view

of non-commutative geometry), the coring $A \otimes_k C$ in Example 4.2 is free as a left A -module. Then the bijectivity of the canonical entwining map (condition (3) in Example 4.2) implies that $A \otimes_k C$ is also free as a right A -module. Thus one could get even closer to a principal extension by considering principal comodules over corings that are free as left and right A -modules. For the purpose of a general exposition in this paper, however, this would be an unnecessary restriction on a coring.

Since a principal comodule M is a projective S -module, it is a flat S -module (thus, in particular ${}_A C$ is a flat module and M is a generator in \mathbf{M}^C by the Galois comodule structure theorem). In fact, the principality of a comodule implies faithful flatness. More precisely one proves the following

Theorem 4.3. *Let M be a principal C -comodule that is faithfully flat as a k -module and set $S = \text{End}^{-C}(M)$. Then M is a faithfully flat left S -module.*

Proof. Consider an epimorphism $f : V \rightarrow W$ of right C -comodules. Since ${}_S M$ is flat, both evaluation maps φ_V and φ_W in Theorem 2.1(1)(c) are isomorphisms in \mathbf{M}^C . Note that

$$\varphi_W^{-1} \circ f \circ \varphi_V = \text{Hom}^{-C}(M, f) \otimes_S M,$$

and thus we obtain an exact sequence of right C -comodule maps

$$\text{Hom}^{-C}(M, V) \otimes_S M \xrightarrow{\text{Hom}^{-C}(M, f) \otimes_S M} \text{Hom}^{-C}(M, W) \otimes_S M \longrightarrow 0.$$

Let $\sigma : M \rightarrow S \otimes_k M$ be a left S -linear splitting of the S -action ${}_M \varrho$. Then the above exact sequence leads to the following commutative diagram with exact top row and split-exact columns

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ \text{Hom}^{-C}(M, V) \otimes_S M & \xrightarrow{\text{Hom}^{-C}(M, f) \otimes_S M} & \text{Hom}^{-C}(M, W) \otimes_S M & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ \text{Hom}^{-C}(M, V) \otimes_S M \varrho & \xrightarrow{\text{Hom}^{-C}(M, V) \otimes_S \sigma} & \text{Hom}^{-C}(M, W) \otimes_S M \varrho & \xrightarrow{\text{Hom}^{-C}(M, W) \otimes_S \sigma} & \\ \uparrow & & \uparrow & & \\ \text{Hom}^{-C}(M, V) \otimes_k M & \xrightarrow{\text{Hom}^{-C}(M, f) \otimes_k M} & \text{Hom}^{-C}(M, W) \otimes_k M & & \end{array}$$

Therefore the map $\text{Hom}^{-C}(M, f) \otimes_k M$ is surjective (for any $x \in \text{Hom}^{-C}(M, W) \otimes_k M$ is of the form $(\text{Hom}^{-C}(M, f) \otimes_k M)(y)$, where y is such that $(\text{Hom}^{-C}(M, f) \otimes_S M)(y) = (\text{Hom}^{-C}(M, W) \otimes_S M \varrho)(x)$). Since M is a faithfully flat k -module, also the map $\text{Hom}^{-C}(M, f) : \text{Hom}^{-C}(M, V) \rightarrow \text{Hom}^{-C}(M, W)$ is surjective. This implies that M is a projective object in \mathbf{M}^C (cf. [9, 18.20]). Thus M is a projective generator in

$\mathbf{M}^{\mathcal{C}}$ and the Galois comodule structure theorem implies that ${}_S M$ is a faithfully flat module. \square

Thus, by the Galois comodule structure theorem, every principal \mathcal{C} -comodule M that is faithfully flat as a k -module is a finitely generated projective generator in $\mathbf{M}^{\mathcal{C}}$, and it induces the category equivalence $- \otimes_S M : \mathbf{M}_S \rightarrow \mathbf{M}^{\mathcal{C}}$.

In case k is a field, one can derive a description of principal comodules which resembles the difficult part of the Schneider theorem. To facilitate this description, note that in parallel to the theory of Galois right comodules one develops the theory of Galois left comodules. Consider a left comodule N of an A -coring \mathcal{C} with the endomorphism ring $T = \text{End}^{\mathcal{C}^-}(N)$. The product in T is given by opposite composition, i.e., $tt' = t' \circ t$. This makes N into a right T -module, with multiplication $nt = t(n)$. N is called a *Galois (left) comodule* provided it is finitely generated projective left A -module and the evaluation map

$$\widehat{\varphi}_{\mathcal{C}} : N \otimes_T \text{Hom}^{\mathcal{C}^-}(N, \mathcal{C}) \rightarrow \mathcal{C}, \quad n \otimes f \mapsto f(n),$$

is an isomorphism of left \mathcal{C} -comodules. Equivalently, N is a Galois left \mathcal{C} -comodule provided ${}_A N$ is finitely generated projective and the left canonical map

$${}_N \text{can} : N \otimes_T {}^* N \rightarrow \mathcal{C}, \quad n \otimes \xi \mapsto \sum n_{(-1)} \xi(n_{(0)}),$$

is an isomorphism of A -corings.

Any Galois right \mathcal{C} -comodule M gives rise to a Galois left \mathcal{C} -comodule. First, since M is a finitely generated projective right A -module, the dual module $M^* = \text{Hom}_{-A}(M, A)$ is a left \mathcal{C} -comodule with the coaction determined by

$$(4.1) \quad \sum \xi(m_{(0)}) m_{(1)} = \sum \xi_{(-1)} \xi_{(0)}(m), \quad \forall \xi \in M^*, m \in M.$$

Explicitly, in terms of a dual basis $\{e^i \in M, \xi^i \in M^*\}_{i=1, \dots, n}$ of M , the left coaction comes out as

$${}^{M^*} \varrho : M^* \rightarrow \mathcal{C} \otimes_A M^*, \quad \xi \mapsto \sum_i \xi(e^i_{(0)}) e^i_{(1)} \otimes \xi^i.$$

This definition of the left \mathcal{C} -coaction also implies the following equality for the elements of a dual basis

$$(4.2) \quad \sum_i e^i_{(0)} \otimes e^i_{(1)} \otimes \xi^i = \sum_i e^i \otimes \xi^i_{(-1)} \otimes \xi^i_{(0)}.$$

Second, there is a ring isomorphism $\Gamma_M : S = \text{End}^{-\mathcal{C}}(M) \rightarrow \text{End}^{\mathcal{C}^-}(M^*)$. Explicitly, for all $s \in S$, $\xi \in M^*$, $t \in \text{End}^{\mathcal{C}^-}(M^*)$ and $m \in M$, the map Γ_M and its inverse Γ_M^{-1}

are given by

$$(4.3) \quad \Gamma_M(s)(\xi) = \sum_i \xi(s(e^i))\xi^i, \quad \Gamma_M^{-1}(t)(m) = \sum_i e^i t(\xi^i)(m).$$

Thus M^* is a right S -module with the multiplication $\xi s = \Gamma_M(s)(\xi) = \xi \circ s$. Note that Γ_M is a restriction of the canonical isomorphism $\text{End}_{-A}(M) \rightarrow \text{End}_{A-}(M^*)$, which we also denote by Γ_M . Third, with this isomorphism of endomorphism rings, and with the help of equation (4.1), the left canonical map comes out as ${}_{M^*}\text{can} : M^* \otimes_S M \rightarrow \mathcal{C}$, ${}_{M^*}\text{can}(\xi \otimes m) = \sum \xi_{(-1)}\xi_{(0)}(m) = \sum \xi(m_{(0)})m_{(1)}$ and thus ${}_{M^*}\text{can} = \text{can}_M$. Since can_M is an isomorphism of corings, so is ${}_{M^*}\text{can}$, and therefore M^* is a Galois left \mathcal{C} -comodule.

This interlude on Galois left comodules allows one to state the following theorem which is a coring origin of [20, Theorem 2.3.4].

Theorem 4.4. *Let k be a field, \mathcal{C} an A -coring and M a right \mathcal{C} -comodule that is finitely generated and projective as a right A -module. Set $S = \text{End}^{-\mathcal{C}}(M)$. View $M^* \otimes_k M$ as a left \mathcal{C} -comodule via ${}^{M^*}\varrho \otimes_k M$. Then the following statements are equivalent:*

(1) \mathcal{C} is a flat left A -module and the map

$$\widetilde{\text{can}}_M : M^* \otimes_k M \rightarrow \mathcal{C}, \quad \xi \otimes m \mapsto \sum \xi(m_{(0)})m_{(1)},$$

is a split epimorphism of left \mathcal{C} -comodules.

(2) M is a principal right \mathcal{C} -comodule.

The proof of Theorem 4.4 makes use of the following

Lemma 4.5. *Let k be a field, \mathcal{C} an A -coring and N a left \mathcal{C} -comodule that is finitely generated and projective as a left A -module. Then for any k -vector space V ,*

$$\text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V) \simeq \text{End}^{\mathcal{C}^-}(N) \otimes_k V,$$

where $N \otimes_k V$ is a left \mathcal{C} -comodule with the coaction ${}^N\varrho \otimes_k V$.

Proof. Since ${}_A N$ is finitely generated and projective, there is a k -linear isomorphism $\theta : \text{End}_{A-}(N) \otimes_k V \rightarrow \text{Hom}_{A-}(N, N \otimes_k V)$. Explicitly, for all $t \in \text{End}_{A-}(N)$, $v \in V$ and $n \in N$, $\theta(t \otimes v)(n) = t(n) \otimes v$. Clearly, if $t \in \text{End}^{\mathcal{C}^-}(N)$, then $\theta(t \otimes v) \in \text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V)$. Thus we only need to check whether the inverse θ^{-1} of θ restricted to $\text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V)$ has the image in $\text{End}^{\mathcal{C}^-}(N) \otimes_k V$.

To write θ^{-1} explicitly, choose a dual basis $\{\xi^i \in N, e^i \in {}^*N\}_{i=1, \dots, l}$ of ${}_A N$ and, for all $f \in \text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V)$ and $n \in N$, write $f(n) = \sum f(n)^{(1)} \otimes f(n)^{(2)} \in N \otimes_k V$.

Then

$$\theta^{-1}(f) = \sum_i e^i(-) f(\xi^i)^{(1)} \otimes f(\xi^i)^{(2)}.$$

Since f is a left \mathcal{C} -comodule map, for all $n \in N$,

$$\begin{aligned} \sum n_{(-1)} \otimes \theta^{-1}(f)(n_{(0)}) &= \sum_i n_{(-1)} \otimes e^i(n_{(0)}) f(\xi^i)^{(1)} \otimes f(\xi^i)^{(2)} \\ &= \sum_i n_{(-1)} \otimes f(e^i(n_{(0)}) \xi^i)^{(1)} \otimes f(e^i(n_{(0)}) \xi^i)^{(2)} \\ &= \sum n_{(-1)} \otimes f(n_{(0)})^{(1)} \otimes f(n_{(0)})^{(2)} \\ &= \sum f(n)^{(1)}_{(-1)} \otimes f(n)^{(1)}_{(0)} \otimes f(n)^{(2)}. \end{aligned}$$

The second equality follows from the left A -linearity of f , the third one is a consequence of the properties of a dual basis, and the last equality results from \mathcal{C} -colinearity of f . On the other hand, again using the properties of a dual basis and the A -linearity of f , we obtain

$$\begin{aligned} ({}^N \varrho \otimes_k V)(\theta^{-1}(f)(n)) &= \sum_i (e^i(n) f(\xi^i)^{(1)})_{(-1)} \otimes (e^i(n) f(\xi^i)^{(1)})_{(0)} \otimes f(\xi^i)^{(2)} \\ &= \sum_i f(e^i(n) \xi^i)^{(1)}_{(-1)} \otimes f(e^i(n) \xi^i)^{(1)}_{(0)} \otimes f(e^i(n) \xi^i)^{(2)} \\ &= \sum f(n)^{(1)}_{(-1)} \otimes f(n)^{(1)}_{(0)} \otimes f(n)^{(2)}. \end{aligned}$$

Thus $\text{Im}(\theta^{-1} |_{\text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V)}) \subseteq \text{End}^{\mathcal{C}^-}(N) \otimes_k V$, hence θ restricts to an isomorphism $\text{End}^{\mathcal{C}^-}(N) \otimes_k V \simeq \text{Hom}^{\mathcal{C}^-}(N, N \otimes_k V)$, as required. \square

Proof of Theorem 4.4. (1) \Rightarrow (2) Lemma 4.5 and the identification of left \mathcal{C} -endomorphisms of M^* with S leads to the isomorphism $\text{Hom}^{\mathcal{C}^-}(M^*, M^* \otimes_k M) \simeq \text{End}^{\mathcal{C}^-}(M^*) \otimes_k M \simeq S \otimes_k M$. Furthermore, $\text{Hom}^{\mathcal{C}^-}(M^*, \mathcal{C}) \simeq {}^*(M^*) \simeq M$. Since \mathcal{C} is a direct summand of $M^* \otimes_k M$ as a left \mathcal{C} -comodule, and $\text{Hom}^{\mathcal{C}^-}(M^*, -)$ is a functor from ${}^{\mathcal{C}}\mathbf{M}$ to ${}_S\mathbf{M}$, the above isomorphisms imply that M is a direct summand of $S \otimes_k M$ as a left S -module. Since k is assumed to be a field, $S \otimes_k M$ is a free S -module, hence M is a projective left S -module.

The first step in the proof that M is a Galois comodule is to consider the following commutative diagram

$$\begin{array}{ccc} M^* \otimes_S \text{Hom}^{\mathcal{C}^-}(M^*, M^* \otimes_k M) & \xrightarrow{\hat{\varphi}_{M^* \otimes_k M}} & M^* \otimes_k M \\ & \searrow \simeq & \nearrow \simeq \\ & M^* \otimes_S S \otimes_k M & \end{array},$$

where the first isomorphism follows from Lemma 4.5 and the discussion above. Thus $\widehat{\varphi}_{M^* \otimes_k M}$ is an isomorphism. Next we can consider the following diagram, which is commutative in all possible directions since $\widehat{\varphi}$ is a natural transformation,

$$\begin{array}{ccc}
 M^* \otimes_S \text{Hom}^{C-}(M^*, M^* \otimes_k M) & \xrightarrow{\widehat{\varphi}_{M^* \otimes_k M}} & M^* \otimes_k M \\
 \uparrow \downarrow M^* \otimes_S \text{Hom}^{C-}(M^*, \widetilde{\text{can}}_M) & & \uparrow \downarrow \widetilde{\text{can}}_M \\
 M^* \otimes_S \text{Hom}^{C-}(M^*, \mathcal{C}) & \xrightarrow{\widehat{\varphi}_{\mathcal{C}}} & \mathcal{C}
 \end{array}$$

The upward pointing arrows are sections of $M^* \otimes_S \text{Hom}^{C-}(M^*, \widetilde{\text{can}}_M)$ and $\widetilde{\text{can}}_M$ respectively. Since $\widehat{\varphi}_{M^* \otimes_k M}$ is an isomorphism, the map $\widehat{\varphi}_{\mathcal{C}}$ is one-to-one and onto (it is a k -linear isomorphism). With the help of identifications $\text{Hom}^{C-}(M^*, \mathcal{C}) \simeq M$ and $M^* \simeq \text{Hom}^{-C}(M, \mathcal{C})$ we can construct yet another commutative diagram

$$\begin{array}{ccc}
 M^* \otimes_S \text{Hom}^{C-}(M^*, \mathcal{C}) & \xrightarrow{\widehat{\varphi}_{\mathcal{C}}} & \mathcal{C} \\
 \searrow \simeq & & \nearrow \varphi_{\mathcal{C}} \\
 & \text{Hom}^{-C}(M, \mathcal{C}) \otimes_S M &
 \end{array}$$

Since $\widehat{\varphi}_{\mathcal{C}}$ is one-to-one and onto, so is $\varphi_{\mathcal{C}}$. By assumption, \mathcal{C} is a flat left A -module, so $\mathbf{M}^{\mathcal{C}}$ is an Abelian category. Since $\varphi_{\mathcal{C}}$ is a one-to-one and onto morphism in $\mathbf{M}^{\mathcal{C}}$, it is an isomorphism. Thus M is a Galois right \mathcal{C} -comodule that is projective over its endomorphism ring, i.e., M is a principal comodule.

(2) \Rightarrow (1) Applying functor $M^* \otimes_S - : {}_S \mathbf{M} \rightarrow {}^{\mathcal{C}} \mathbf{M}$ to a left S -linear section of the multiplication $S \otimes_k M \rightarrow M$, one obtains a left \mathcal{C} -comodule section of the canonical surjection $M^* \otimes_k M \rightarrow M^* \otimes_S M$. Composed with an A -coring (hence also a left \mathcal{C} -comodule) map $\text{can}_M^{-1} : \mathcal{C} \rightarrow M^* \otimes_S M$ this gives the required section of $\widetilde{\text{can}}_M$. \square

Theorem 4.4 leads to the main geometric result of this section, namely to a condition for an entwining structure to give rise to a principal extension or a non-commutative principal fibre bundle. First recall that a coalgebra C is said to be *coseparable* provided the coproduct has a retraction in the category of C -bicomodules. Equivalently, C is a coseparable coalgebra if there exists a *cointegral*, i.e., a k -module map $\delta : C \otimes_k C \rightarrow k$ such that $\delta \circ \Delta_C = \varepsilon_C$ and, for all $c, c' \in C$,

$$\sum c_{(1)} \delta(c_{(2)} \otimes c') = \sum \delta(c \otimes c'_{(1)}) c'_{(2)}.$$

This equality is known as the *colinearity* of the cointegral δ . Any cosemisimple coalgebra over an algebraically closed field is coseparable (cf. [20, Remark 2.5.3]).

Theorem 4.6. *Let k be a field and $(A, C)_\psi$ an entwining structure such that the map ψ is bijective. Suppose that $e \in C$ is a group-like element and view A as a right C -comodule with the coaction $\varrho^A : A \rightarrow A \otimes_k C$, $a \mapsto \psi(e \otimes a)$. If C is a coseparable coalgebra and the (lifted) canonical map*

$$\widetilde{\text{can}} : A \otimes_k A \rightarrow A \otimes_k C, \quad a \otimes a' \mapsto a \varrho^A(a')$$

is surjective, then A is a principal C -extension of the coinvariants $S = A^{\text{co}C}$.

Proof. The strategy for the proof is first to use Theorem 4.4 (with $M = A$) to show that A is a principal comodule of the coring $\mathcal{C} = A \otimes_k C$ corresponding to $(A, C)_\psi$, and then to show that the projectivity of ${}_S A$ implies the C -equivariant projectivity (cf. [20, Proposition 2.5.4]).

Following Theorem 4.4 we need to construct a left \mathcal{C} -comodule splitting of $\widetilde{\text{can}}$. The right C -coaction ϱ^A can be understood as a right \mathcal{C} -coaction corresponding to a group-like element $g = 1 \otimes e \in \mathcal{C}$, i.e., $\varrho^A(a) = \psi(e \otimes a) = (1 \otimes e)a = ga$ (cf. proof of Example 4.2). Thus the left \mathcal{C} -coaction of $A^* \simeq A$ comes out as $a \mapsto ag = a \otimes e$. Since ψ is bijective, this left \mathcal{C} -coaction gives rise to a left C -coaction ${}^A \varrho(a) = \psi^{-1}(a \otimes e) := \sum a_{(-1)} \otimes a_{(0)}$. In view of the isomorphism $\text{Hom}_{A-}(A \otimes_k C, A \otimes_k A) \simeq \text{Hom}_k(C, A \otimes_k A)$, any left \mathcal{C} -comodule map $f : A \otimes_k C \rightarrow A \otimes_k A$ can be identified with a k -linear map $\hat{f} : C \rightarrow A \otimes_k A$ such that, for all $c \in C$, writing $\hat{f}(c) := \sum \hat{f}(c)^{(1)} \otimes \hat{f}(c)^{(2)} \in A \otimes_k A$,

$$\sum \psi(c_{(1)} \otimes \hat{f}(c_{(2)}))^{(1)} \otimes \hat{f}(c)^{(2)} = \sum \hat{f}(c)^{(1)} \otimes e \otimes \hat{f}(c)^{(2)} \in A \otimes_k C \otimes_k A.$$

Applying $\psi^{-1} \otimes_k A$ we thus, equivalently, obtain the condition

$$(C \otimes_k \hat{f}) \circ \Delta_C = ({}^A \varrho \otimes_k A) \circ \hat{f}. \quad (*)$$

Since $\widetilde{\text{can}}$ is surjective, it has a k -linear section $\tau : A \otimes_k C \rightarrow A \otimes_k A$. Define $\hat{\tau} : C \rightarrow A \otimes_k A$ by $\hat{\tau}(c) = \tau(1 \otimes c)$. Let δ be a cointegral of C and define a k -linear map $\hat{\kappa} : C \rightarrow A \otimes_k A$, by

$$\hat{\kappa} = (\delta \otimes_k A \otimes_k A) \circ (C \otimes_k {}^A \varrho \otimes_k A) \circ (C \otimes_k \hat{\tau}) \circ \Delta_C.$$

Using the colinearity of δ one easily checks that $\hat{\kappa}$ has the property $(*)$. Therefore the map $\kappa : A \otimes_k C \rightarrow A \otimes_k A$, $\kappa(a \otimes c) = a \hat{\kappa}(c)$ is a left \mathcal{C} -comodule morphism. We aim to prove that κ is a section of $\widetilde{\text{can}}$.

To this end, first introduce the α -notation for an entwining map and its inverse, i.e., for all $a \in A$, $c \in C$, write

$$\psi(c \otimes a) = \sum_{\alpha} a_{\alpha} \otimes c^{\alpha}, \quad \psi^{-1}(a \otimes c) = \sum_A c_A \otimes a^A.$$

In this notation the left and right pentagon conditions in the bow-tie diagram read respectively,

$$\psi(c \otimes aa') = \sum_{\alpha, \beta} a_\alpha a'_\beta \otimes c^{\alpha\beta}, \quad \sum_{\alpha} a_\alpha \otimes c^\alpha_{(1)} \otimes c^\alpha_{(2)} = \sum_{\alpha, \beta} a_{\beta\alpha} \otimes c_{(1)}^\alpha \otimes c_{(2)}^\beta.$$

Since ψ^{-1} is the inverse of ψ ,

$$a \otimes c = \sum_{\alpha, A} a_\alpha^A \otimes c_A^\alpha.$$

Second, write $\hat{\tau}(c) = \sum c^{(1)} \otimes c^{(2)} \in A \otimes_k A$, so that the map κ explicitly reads

$$\kappa(a \otimes c) = \sum a \delta(c_{(1)} \otimes c_{(2)}^{(1)}_{(-1)}) c_{(2)}^{(1)}_{(0)} \otimes c_{(2)}^{(2)}.$$

Note that, since $\hat{\tau}$ is obtained from a k -linear section of $\widetilde{\text{can}}$, for all $c \in C$,

$$\sum c^{(1)} c^{(2)}_{(0)} \otimes c^{(2)}_{(1)} = 1 \otimes c. \quad (**)$$

In view of the definitions of left and right C -coactions, for all $c \in C$,

$$\sum c^{(1)}_{(-1)} \otimes c^{(1)}_{(0)} c^{(2)}_{(0)} \otimes c^{(2)}_{(1)} = \sum \psi^{-1}(c^{(1)} \otimes e) \psi(e \otimes c^{(2)}) = \sum_{\alpha, A} e_A \otimes c^{(1)A}_\alpha c^{(2)}_\alpha \otimes e^\alpha.$$

Apply $\psi \otimes C$ to this identity and compute

$$\begin{aligned} \sum \psi(c^{(1)}_{(-1)} \otimes c^{(1)}_{(0)} c^{(2)}_{(0)} \otimes c^{(2)}_{(1)}) &= \sum_{\alpha, A} \psi(e_A \otimes c^{(1)A}_\alpha c^{(2)}_\alpha \otimes e^\alpha) \\ &= \sum_{\alpha, \beta, \gamma, A} c^{(1)A}_\gamma c^{(2)}_{\alpha\beta} \otimes e^\gamma_A \otimes e^\alpha \\ &= \sum_{\alpha, \beta} c^{(1)} c^{(2)}_{\alpha\beta} \otimes e^\beta \otimes e^\alpha \\ &= \sum_{\alpha} c^{(1)} c^{(2)}_\alpha \otimes e^\alpha_{(1)} \otimes e^\alpha_{(2)} \\ &= \sum c^{(1)} c^{(2)}_{(0)} \otimes c^{(2)}_{(1)} \otimes c^{(2)}_{(2)} \\ &= \sum 1 \otimes c_{(1)} \otimes c_{(2)}. \end{aligned}$$

The second equality follows from the left pentagon in the bow-tie diagram, while the third one is the consequence of the fact that ψ^{-1} is the inverse of ψ . Next, observing that e is a group-like element and using the right pentagon in the bow-tie diagram we arrive at the fourth equality. The remaining two equalities follow from the definition of the right C -coaction and from equation (**). The left triangle in the bow-tie diagram implies that $\psi^{-1}(1 \otimes c) = c \otimes 1$, thus applying ψ^{-1} to the equality just derived, we conclude that

$$\sum c^{(1)}_{(-1)} \otimes c^{(1)}_{(0)} c^{(2)}_{(0)} \otimes c^{(2)}_{(1)} = \sum c_{(1)} \otimes 1 \otimes c_{(2)}. \quad (***)$$

Now equation $(***)$ facilitates the following computation

$$\begin{aligned} (\widetilde{\text{can}} \circ \kappa)(a \otimes c) &= \sum a \delta(c_{(1)} \otimes c_{(2)}^{(1)}_{(-1)}) c_{(2)}^{(1)}_{(0)} c_{(2)}^{(2)}_{(0)} \otimes c_{(2)}^{(2)}_{(1)} \\ &= \sum a \delta(c_{(1)} \otimes c_{(2)}) \otimes c_{(3)} = \sum a \otimes \varepsilon_C(c_{(1)}) c_{(2)} = a \otimes c, \end{aligned}$$

where the penultimate equality is the consequence of the fact that δ is a cointegral. Thus we have proven that κ is a left \mathcal{C} -colinear section of the canonical map $\widetilde{\text{can}}$ so that A is a principal right \mathcal{C} -comodule by Theorem 4.4. This means in particular that A is a Galois C -extension of S , i.e., condition (1) in Example 4.2 is satisfied. Furthermore, the uniqueness of the canonical entwining map (cf. [6, Theorem 2.7]) implies that ψ is the canonical entwining map, hence it is bijective as required for condition (3) in Example 4.2. Obviously, condition (4) in Example 4.2 is also satisfied. Thus we only need to construct a section satisfying condition (2) in Example 4.2.

Since A is a principal \mathcal{C} -comodule it is a projective left S -module, hence there is a left S -module section $\tilde{\sigma} : A \rightarrow S \otimes_k A$ of the product. Using the cointegral δ , construct the map

$$\sigma : A \rightarrow S \otimes_k A, \quad \sigma = (S \otimes_k A \otimes_k \delta) \circ (S \otimes_k \varrho^A \otimes_k C) \circ (\tilde{\sigma} \otimes_k C) \circ \varrho^A.$$

Since σ is a composition of S -module maps it is a left S -module map. Using the colinearity of a cointegral, one easily shows that σ is a right C -colinear map. To show that σ is a section of the product map, we denote $\tilde{\sigma}(a) = \sum a^{(1)} \otimes a^{(2)} \in S \otimes_k A$, so that the map σ explicitly reads,

$$\sigma(a) = \sum a_{(0)}^{(1)} \otimes a_{(0)}^{(2)}_{(0)} \delta(a_{(0)}^{(2)}_{(1)} \otimes a_{(1)}).$$

Remember that $\tilde{\sigma}$ is a section of the product map ${}_A \varrho : S \otimes_k A \rightarrow A$, $s \otimes a \rightarrow sa$, so that for all $a \in A$, $\sum a^{(1)} a^{(2)} = a$. Remember also that ϱ^A is a left S -linear map (S is the \mathcal{C} -endomorphism ring of A !). With these facts at hand we can compute

$$\begin{aligned} ({}_A \varrho \circ \sigma)(a) &= \sum a_{(0)}^{(1)} a_{(0)}^{(2)}_{(0)} \delta(a_{(0)}^{(2)}_{(1)} \otimes a_{(1)}) \\ &= \sum (a_{(0)}^{(1)} a_{(0)}^{(2)}_{(0)}) \delta((a_{(0)}^{(1)} a_{(0)}^{(2)}_{(0)})_{(1)} \otimes a_{(1)}) \\ &= \sum a_{(0)} \delta(a_{(1)} \otimes a_{(2)}) = a. \end{aligned}$$

Therefore, σ is a right C -colinear, left S -linear section of the product map, so that also condition (2) in Example 4.2 is satisfied, hence A is a principal C -extension of S . \square

Since any comodule of a coseparable coalgebra over a field is an injective comodule, Theorem 4.6 can be understood as an entwining structure version of the ‘difficult part’ of Schneider’s structure theorem [21, Theorem I]. As a special case one obtains

Corollary 4.7. [20, Theorem 2.5.7] *Let k be a field, H be a Hopf algebra with a bijective antipode and let A be a right H -comodule algebra with a coaction $\bar{\varrho}^A : A \rightarrow A \otimes_k H$. Let C be a right H -module and a coalgebra quotient of C via a surjection $\pi : H \rightarrow C$. View A as a right C -comodule via the induced coaction $\varrho^A = (A \otimes_k \pi) \circ \bar{\varrho}^A$. Suppose that C is a coseparable coalgebra (or that k is algebraically closed and C is a cosemisimple coalgebra). If the (lifted) canonical map $\widetilde{\text{can}} : A \otimes_k A \rightarrow A \otimes_k C$, $a \otimes a' \mapsto a \varrho^A(a')$ is surjective, then A is a principal C -extension of the coinvariants $S = A^{\text{co}C}$.*

Proof. This is an example of a Doi-Koppinen entwining structure $(A, C)_\psi$ with $\psi : c \otimes a \mapsto \sum a_{(0)} \otimes ca_{(1)}$, where $\bar{\varrho}^A(a) = \sum a_{(0)} \otimes a_{(1)} \in A \otimes_k H$. Since 1_H is a group-like element in H , $e = \pi(1_H)$ is a group-like element in C . Note that

$$\psi(e \otimes a) = \sum a_{(0)} \otimes \pi(1_H) a_{(1)} = \sum a_{(0)} \otimes \pi(a_{(1)}) = \varrho^A(a),$$

for π is a right H -module map. Thus the C -coaction on A has the required form. The map ψ is bijective with the inverse $\psi^{-1} : A \otimes_k C \rightarrow C \otimes_k A$, $a \otimes c \mapsto \sum c S^{-1}(a_{(1)}) \otimes a_{(0)}$, where S is the antipode in H . Any cosemisimple coalgebra over an algebraically closed field is coseparable, thus in either case all the assumptions of Theorem 4.6 are satisfied and the assertion follows. \square

An explicit, very important and geometrically interesting example of Galois corings (principal extensions) of the type described in Corollary 4.7 has been recently constructed in [1]. In this example H is the Hopf algebra of functions on the quantum group $U_q(4)$, the algebra A is the algebra of functions on the quantum 7-sphere S_q^7 , the induced coalgebra C is the coalgebra of functions on the quantum group $SU_q(2)$. Finally, the coinvariant algebra S is the algebra of functions on the ‘‘Etruscan’’ quantum 4-sphere Σ_q^4 introduced in [2].

5. MORPHISMS OF CORINGS AND INDUCED GALOIS AND PRINCIPAL COMODULES

It frequently happens that there is a pair of corings related by a coring morphism, and one can prove that a comodule of one of these corings is, respectively, a Galois or a principal comodule. A question then arises, whether the induced comodule is also a Galois or a principal comodule. This is the main subject of the present section.

Lemma 5.1. *Let $(\gamma : \alpha) : (C : A) \rightarrow (D : B)$ be a morphism of corings. Suppose that M is a Galois right C -comodule and let $S = \text{End}^{-C}(M)$. If M is a flat left S -module*

then for any $N \in \mathbf{M}^{\mathcal{D}}$, the map

$$\vartheta_{M,N} : \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N) \otimes_S M \rightarrow N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}), \quad f \otimes m \mapsto \sum f(m_{(0)} \otimes 1_B) \otimes m_{(1)}$$

is an isomorphism of k -modules. This isomorphism is natural in N , i.e., it is an isomorphism of functors $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -) \otimes_S M \simeq - \square_{\mathcal{C}}(B \otimes_A \mathcal{C})$.

Proof. Since M is a Galois comodule and ${}_S M$ is flat, Theorem 2.1(1) implies that \mathcal{C} is a flat left A -module. Thus $- \square_{\mathcal{D}}(B \otimes_A \mathcal{C})$ is a functor $\mathbf{M}^{\mathcal{D}} \rightarrow \mathbf{M}^{\mathcal{C}}$ and, consequently, $N \square_{\mathcal{D}}(B \otimes_A \mathcal{C})$ is a right \mathcal{C} -comodule. Therefore Theorem 2.1(1) again implies that the evaluation map

$$\varphi_{N \square_{\mathcal{D}}(B \otimes_A \mathcal{C})} : \text{Hom}^{-\mathcal{C}}(M, N \square_{\mathcal{D}}(B \otimes_A \mathcal{C})) \otimes_S M \rightarrow N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}), \quad f \otimes m \mapsto f(m)$$

is an isomorphism of right \mathcal{C} -comodules. Combining $\varphi_{N \square_{\mathcal{D}}(B \otimes_A \mathcal{C})}$ with the general hom-tensor relation isomorphism of k -modules $\text{Hom}^{-\mathcal{C}}(M, N \square_{\mathcal{D}}(B \otimes_A \mathcal{C})) \simeq \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N)$, we obtain the desired isomorphism $\vartheta_{M,N}$.

To verify the naturality of maps $\vartheta_{M,N}$ we need to take any morphism $g : N \rightarrow N'$ in $\mathbf{M}^{\mathcal{D}}$ and establish that the following diagram

$$\begin{array}{ccc} \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N) \otimes_S M & \xrightarrow{\vartheta_{M,N}} & N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \\ \text{Hom}^{-\mathcal{D}}(M \otimes_A B, g) \otimes_S M \downarrow & & \downarrow g \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \\ \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N') \otimes_S M & \xrightarrow{\vartheta_{M,N'}} & N' \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \end{array}$$

is commutative. This follows immediately from the definition of $\vartheta_{M,N}$. \square

Theorem 5.2. *Let $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a morphism of corings. Suppose that M is a Galois right \mathcal{C} -comodule and let $S = \text{End}^{-\mathcal{C}}(M)$ and $T = \text{End}^{-\mathcal{D}}(M \otimes_A B)$. If M is a faithfully flat left S -module then the following statements are equivalent.*

- (1) \mathcal{D} is a flat left B -module and $B \otimes_A \mathcal{C}$ is a faithfully coflat left \mathcal{D} -comodule.
- (2) $M \otimes_A B$ is a Galois right \mathcal{D} -comodule and $M \otimes_A B$ is a faithfully flat left T -module.
- (3) \mathcal{D} is a flat left B -module and $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -) : \mathbf{M}^{\mathcal{D}} \rightarrow \mathbf{M}_T$ is an equivalence with the inverse $- \otimes_T (M \otimes_A B) : \mathbf{M}_T \rightarrow \mathbf{M}^{\mathcal{D}}$.
- (4) \mathcal{D} is a flat left B -module and $M \otimes_A B$ is a projective generator in $\mathbf{M}^{\mathcal{D}}$.

Furthermore, if B is a quasi-Frobenius (QF) ring then the above statements are equivalent to

- (5) \mathcal{D} is a flat left B -module and $B \otimes_A \mathcal{C}$ is an injective cogenerator in ${}^{\mathcal{D}}\mathbf{M}$.

Proof. Clearly, the equivalences $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ are contained in the Galois comodule structure theorem. The equivalence of (1) and (5) (in the case of a QF ring B) follows from the description of faithfully coflat comodules for corings over QF rings (cf. [9, 21.9(2)(ii)]). Thus it suffices to show that the statement (1) is equivalent to the statement (3).

$(3) \Rightarrow (1)$ Since the functor $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -)$ is an equivalence of Abelian categories, it reflects and preserves exact sequences. Furthermore, since ${}_S M$ is a faithfully flat module, the composite functor $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -) \otimes_S M$ preserves and reflects exact sequences. By Lemma 5.1 the functor $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -) \otimes_S M$ is naturally isomorphic to the functor $-\square_{\mathcal{D}}(B \otimes_A \mathcal{C})$, thus the latter also preserves and reflects exact sequences. Hence $B \otimes_A \mathcal{C}$ is a faithfully coflat left \mathcal{D} -comodule.

$(1) \Rightarrow (3)$ Take any right T -module W and any right \mathcal{D} -comodule N . Since the functor $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -)$ is right adjoint to the functor $-\otimes_T (M \otimes_A B)$, there is an isomorphism

$$\text{Hom}_{-T}(W, \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N)) \rightarrow \text{Hom}^{-\mathcal{D}}(W \otimes_T M \otimes_A B, N), \quad f \mapsto F_f.$$

Explicitly, $F_f(w \otimes m \otimes b) = f(w)(m \otimes b)$. We need to show that f is an isomorphism if and only if F_f is an isomorphism.

First note that the map $\vartheta_{M, M \otimes_A B} : T \otimes_S M = \text{End}^{-\mathcal{D}}(M \otimes_A B) \otimes_S M \rightarrow (M \otimes_A B) \square_{\mathcal{D}}(B \otimes_A \mathcal{C})$ in Lemma 5.1 is an isomorphism of left T -modules. Indeed, take any $t, f \in T$ and $m \in M$ and compute

$$\begin{aligned} \vartheta_{M, M \otimes_A B}(tf \otimes m) &= \sum tf(m_{(0)} \otimes 1_B) \otimes m_{(1)} \\ &= \sum t(f(m_{(0)} \otimes 1_B)) \otimes m_{(1)} = t\vartheta_{M, M \otimes_A B}(f \otimes m), \end{aligned}$$

as required. Therefore we can construct an isomorphism

$$\theta : W \otimes_S M \rightarrow (W \otimes_T (M \otimes_A B)) \square_{\mathcal{D}}(B \otimes_A \mathcal{C})$$

as a composition

$$\begin{aligned} W \otimes_S M &\xrightarrow{\simeq} M \otimes_T T \otimes_S M \xrightarrow{W \otimes_T \vartheta_{M, M \otimes_A B}} W \otimes_T ((M \otimes_A B) \square_{\mathcal{D}}(B \otimes_A \mathcal{C})) \\ &\xrightarrow{\simeq} (W \otimes_T (M \otimes_A B)) \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \end{aligned}$$

The last isomorphism is the consequence of the fact that $B \otimes_A \mathcal{C}$ is a (faithfully) coflat left \mathcal{D} -comodule. In this way we are led to the following commutative diagram

$$\begin{array}{ccc} W \otimes_S M & \xrightarrow{\theta} & (W \otimes_T (M \otimes_A B)) \square_{\mathcal{D}} (B \otimes_A \mathcal{C}) \\ f \otimes_S M \downarrow & & \downarrow F_f \square_{\mathcal{D}} (B \otimes_A \mathcal{C}) \\ \text{Hom}^{-\mathcal{D}}(M \otimes_A B, N) \otimes_S M & \xrightarrow{\vartheta_{M,N}} & N \square_{\mathcal{D}} (B \otimes_A \mathcal{C}). \end{array}$$

Since the rows are isomorphisms, M is a faithfully flat left S -module and $B \otimes_A \mathcal{C}$ is a faithfully coflat left \mathcal{D} -comodule, the map f is an isomorphism if and only if F_f is an isomorphism. Thus $\text{Hom}^{-\mathcal{D}}(M \otimes_A B, -) : \mathbf{M}^{\mathcal{D}} \rightarrow M_T$ is an equivalence as required.

□

The following lemma is an immediate consequence of the definition of a faithfully coflat comodule, and gives one a criterion of the faithful coflatness.

Lemma 5.3. *Let $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a morphism of corings. Suppose that:*

- (a) \mathcal{D} is flat as a left B -module;
- (b) $B \otimes_A \mathcal{C}$ is a coflat left \mathcal{D} -comodule;
- (c) the induced map $\tilde{\gamma} : B \otimes_A \mathcal{C} \rightarrow \mathcal{D}$, $b \otimes c \mapsto b\gamma(c)$ is a split epimorphism in $\mathbf{M}^{\mathcal{D}}$.

Then $B \otimes_A \mathcal{C}$ is a faithfully coflat left \mathcal{D} -comodule.

Proof. Condition (c) implies that $N \square_{\mathcal{D}} (B \otimes_A \mathcal{C}) \neq 0$ for all nonzero $N \in \mathbf{M}^{\mathcal{D}}$, and thus the assertion follows from [9, 21.7]. □

The criterion of faithful flatness in Lemma 5.3 assures that the principality of a comodule is carried over to the induced comodule.

Theorem 5.4. *Let $(\gamma : \alpha) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a morphism of corings such that the induced map $\tilde{\gamma} : B \otimes_A \mathcal{C} \rightarrow \mathcal{D}$, $b \otimes c \mapsto b\gamma(c)$ is a split epimorphism in $\mathbf{M}^{\mathcal{D}}$. Suppose that M is a principal \mathcal{C} -comodule that is faithfully flat as a k -module, \mathcal{D} is flat as a left B -module and $B \otimes_A \mathcal{C}$ is a coflat left \mathcal{D} -comodule. Then $M \otimes_A B$ is a principal \mathcal{D} -comodule.*

Proof. Let $S = \text{End}^{-\mathcal{C}}(M)$. Since $\tilde{\gamma}$ is a split epimorphism in $\mathbf{M}^{\mathcal{D}}$, there exists a right \mathcal{D} -comodule V such that

$$\mathcal{D} \oplus V \simeq B \otimes_A \mathcal{C}.$$

Since the cotensor product commutes with the colimits (cf. [9, 21.3(3)]), the above isomorphism induces the isomorphism of left T -modules

$$(M \otimes_A B) \square_{\mathcal{D}} \mathcal{D} \oplus (M \otimes_A B) \square_{\mathcal{D}} V \simeq (M \otimes_A B) \square_{\mathcal{D}} (B \otimes_A \mathcal{C}).$$

Note that $(M \otimes_A B) \square_{\mathcal{D}} \mathcal{D} \simeq M \otimes_A B$ and

$$(M \otimes_A B) \square_{\mathcal{D}} (B \otimes_A \mathcal{C}) \simeq \mathrm{Hom}^{-\mathcal{D}}(M \otimes_A B, M \otimes_A B) \otimes_S M = T \otimes_S M,$$

by Lemma 5.1. This is again an isomorphism of left T -modules (compare the proof of Theorem 5.2 (1) \Rightarrow (3)). Since M is a projective left S -module, the induced module $T \otimes_S M$ is a projective left T -module. Thus ${}_T(M \otimes_A B)$ is a direct summand of a projective module, hence a projective module.

In view of Lemma 5.3 and Theorem 4.3, the hypothesis (1) in Theorem 5.2 is satisfied, hence $M \otimes_A B$ is a Galois \mathcal{D} -comodule. Since it is also a projective T -module, it is a principal comodule as claimed. \square

The q -deformed second Hopf fibration of [1], [2] as described at the end of Section 4 is an example of induction of principal comodules provided by Theorem 5.4.

6. DUALITY AND ASSOCIATED MODULES (NON-COMMUTATIVE VECTOR BUNDLES)

The following theorem describes a remarkable duality of Galois comodules.

Theorem 6.1.

- (1) *Let M be a Galois right \mathcal{C} -comodule and let $S = \mathrm{End}^{-\mathcal{C}}(M)$. If M is flat as a left S -module, then for any right \mathcal{C} -comodule W ,*

$$\mathrm{Hom}^{-\mathcal{C}}(W, M) \simeq \mathrm{Hom}_{-S}(\mathrm{Hom}^{-\mathcal{C}}(M, W), S) = (\mathrm{Hom}^{-\mathcal{C}}(M, W))^*,$$

as left S -modules.

- (2) *Let N be a Galois left \mathcal{C} -comodule and let $T = \mathrm{End}^{\mathcal{C}-}(N)$. If N is flat as a right T -module, then for any left \mathcal{C} -comodule V ,*

$$\mathrm{Hom}^{\mathcal{C}-}(V, N) \simeq \mathrm{Hom}_{T-}(\mathrm{Hom}^{\mathcal{C}-}(N, V), T) = {}^*(\mathrm{Hom}^{\mathcal{C}-}(N, V)),$$

as right T -modules.

Proof. We only prove assertion (1), since (2) will follow by the right-left symmetry. Since ${}_S M$ is flat, the first part of the Galois comodule structure theorem, Theorem 2.1, implies that $W \simeq \mathrm{Hom}^{-\mathcal{C}}(M, W) \otimes_S M$. Apply $\mathrm{Hom}^{-\mathcal{C}}(-, M)$ to this isomorphism to deduce that

$$\mathrm{Hom}^{-\mathcal{C}}(W, M) \simeq \mathrm{Hom}^{-\mathcal{C}}(\mathrm{Hom}^{-\mathcal{C}}(M, W) \otimes_S M, M).$$

Now, the hom-tensor relations [9, 18.10(2)] imply that

$$\mathrm{Hom}^{-\mathcal{C}}(\mathrm{Hom}^{-\mathcal{C}}(M, W) \otimes_S M, M) \simeq \mathrm{Hom}_{-S}(\mathrm{Hom}^{-\mathcal{C}}(M, W), \mathrm{Hom}^{-\mathcal{C}}(M, M)),$$

i.e.,

$$\mathrm{Hom}^{-\mathcal{C}}(W, M) \simeq \mathrm{Hom}_{-S}(\mathrm{Hom}^{-\mathcal{C}}(M, W), S) = (\mathrm{Hom}^{-\mathcal{C}}(M, W))^*,$$

as required. Note that all the maps in this chain of isomorphism are maps of left S -modules. \square

Since a dual of a Galois right \mathcal{C} -comodule is a Galois left \mathcal{C} -comodule and the endomorphism rings of these comodules are mutually isomorphic, Theorem 6.1 leads immediately to the following

Corollary 6.2. *Let M be a Galois right \mathcal{C} -comodule and let $S = \mathrm{End}^{-\mathcal{C}}(M)$. If M^* is flat as a right S -module, then for any left \mathcal{C} -comodule V ,*

$$\mathrm{Hom}^{\mathcal{C}-}(V, M^*) \simeq \mathrm{Hom}_{S-}(\mathrm{Hom}^{\mathcal{C}-}(M^*, V), S) = {}^*(\mathrm{Hom}^{\mathcal{C}-}(M^*, V)),$$

as right S -modules.

Recall that for a right \mathcal{C} -comodule M that is finitely generated and projective as a right A -module, $\mathrm{Hom}^{-\mathcal{C}}(M, W) \simeq W \square_{\mathcal{C}} M^*$, for any right \mathcal{C} -comodule W (cf. [9, 21.8]). Explicitly $f \mapsto \sum_i f(e^i) \otimes \xi^i$, where $\{e^i \in M, \xi^i \in M^*\}_{i=1, \dots, n}$ is a dual basis of M . Similar isomorphism holds for left \mathcal{C} -comodules. Taking these isomorphisms into account, one obtains the following immediate consequence of Theorem 6.1 and Corollary 6.2

Corollary 6.3. *With the notation and assumptions as in Theorem 6.1,*

$$\mathrm{Hom}^{-\mathcal{C}}(W, M) \simeq (W \square_{\mathcal{C}} M^*)^*, \quad \mathrm{Hom}^{\mathcal{C}-}(V, N) \simeq {}^*(N \square_{\mathcal{C}} V).$$

Furthermore, for any Galois right \mathcal{C} -comodule M such that M_S^* is flat, and for any left \mathcal{C} -comodule V

$${}^*(M \square_{\mathcal{C}} V) \simeq \mathrm{Hom}^{\mathcal{C}-}(V, M^*).$$

In particular, in the case of a Galois coring (i.e., when A is a Galois comodule), $A^* \simeq {}^*A \simeq A$, and hence some of the stars can be removed in Corollary 6.3, thus leading to

Corollary 6.4. *Let \mathcal{C} be a Galois A -coring, $g = \varrho^A(1)$ be the corresponding group-like element. Then endomorphisms $S = \mathrm{End}^{-\mathcal{C}}(A)$ come out as $S = \{s \in A \mid sg = gs\}$ (cf. Example 4.2).*

If A is flat as a left S -module, then, for any right \mathcal{C} -comodule W , there is an isomorphism of left S -modules

$$(W_g^{\text{co } \mathcal{C}})^* = \text{Hom}_{-S}(W \square_{\mathcal{C}} A, S) \simeq \text{Hom}^{-\mathcal{C}}(W, A),$$

where $W_g^{\text{co } \mathcal{C}} = \{w \in W \mid \varrho^W(w) = w \otimes g\}$ is a right S -module of g -coinvariants of a right \mathcal{C} -comodule W . Note that

$$\text{Hom}^{-\mathcal{C}}(W, A) = \{f \in \text{Hom}_{-A}(W, A) \mid \forall w \in W, \sum f(w_{(0)})w_{(1)} = gf(w)\}.$$

If A is flat as a right S -module, then, for any left \mathcal{C} -comodule V , there is an isomorphism of right S -modules

$$*({}^{\text{co } \mathcal{C}}V_g) = \text{Hom}_{S-}(A \square_{\mathcal{C}} V, S) \simeq \text{Hom}^{\mathcal{C}-}(V, A),$$

where ${}^{\text{co } \mathcal{C}}V_g = \{v \in V \mid {}^V\varrho(v) = g \otimes v\}$ is a left S -module of g -coinvariants of a left \mathcal{C} -comodule V . Note that

$$\text{Hom}^{\mathcal{C}-}(V, A) = \{f \in \text{Hom}_{A-}(V, A) \mid \forall v \in V, \sum v_{(-1)}f(v_{(0)}) = f(v)g\}.$$

Example 6.5. As an example for Corollary 6.4, take a coalgebra-Galois \mathcal{C} -extension $S \subseteq A$ over a field k , i.e., A , C and S are as in Example 4.2 but only the condition Example 4.2 (1) is required to hold. As recalled in the proof of Example 4.2, $\mathcal{C} = A \otimes_k C$ is then an A -coring, and A is a Galois comodule (hence $A \otimes_k C$ is a Galois coring). The group-like element is $\varrho^A(1) = \sum 1_{(0)} \otimes 1_{(1)} \in A \otimes_k C$. Take a left \mathcal{C} -comodule U . Then $V = A \otimes_k U$ is a left \mathcal{C} -comodule with the coaction $a \otimes u \mapsto \sum a \otimes u_{(-1)} \otimes u_{(0)} \in A \otimes_k C \otimes_k U \simeq A \otimes_k C \otimes_A A \otimes_k U$, and $\text{Hom}^{\mathcal{C}-}(V, A) = \text{Hom}_{\psi}(U, A)$, where

$$\text{Hom}_{\psi}(U, A) = \{f \in \text{Hom}_k(U, A) \mid \forall u \in U, \sum \psi(u_{(-1)} \otimes f(u_{(0)})) = \sum f(u)1_{(0)} \otimes 1_{(1)}\}.$$

Here ψ is the canonical entwining map (cf. Example 4.2(3)). On the other hand ${}^{\text{co } \mathcal{C}}V_g = A \square_{\mathcal{C}} U$. Thus, if A_S is flat, Corollary 6.4 implies that $\text{Hom}^{S-}(A \square_{\mathcal{C}} U, S) \simeq \text{Hom}_{\psi}(U, A)$ as right S -modules and we obtain (a part of) [4, Theorem 4.3].

Now take a right \mathcal{C} -comodule X . Then $W = X \otimes_k A$ is a right \mathcal{C} -comodule with the coaction $\varrho^W : x \otimes a \mapsto \sum x_{(0)} \otimes \psi(x_{(1)} \otimes a)$. In this case

$$\text{Hom}^{-\mathcal{C}}(W, A) \simeq \text{Hom}^{-C}(X, A)$$

and

$$W_g^{\text{co } \mathcal{C}} = (X \otimes_k A)_0 := \{x \otimes a \in X \otimes_k A \mid \sum x_{(0)} \otimes \psi(x_{(1)} \otimes a) = \sum x \otimes a 1_{(0)} \otimes 1_{(1)}\}.$$

Then, if A is a flat left S -module, Corollary 6.4 yields the isomorphism of left S -modules $\text{Hom}_{-S}((X \otimes_k A)_0, S) \simeq \text{Hom}^{-C}(X, A)$ in [4, Theorem 5.4].

As explained in [4], both $A \square_C U$ and $(X \otimes_k A)_0$ in Example 6.5 have the non-commutative geometric meaning of fibre bundles associated to non-commutative principal bundles. The isomorphisms described in this example generalise the classical correspondence between sections of a fibre bundle and covariant (tensorial) functions on the principal bundle with values in the fibre (functions of type ρ). Thus Theorem 6.1 (and its corollaries) can be understood as an algebraic origin of this deep geometric fact.

One of the main motivations for introducing principal extensions in [7] is the observation that if A is a principal C -extension and X is a finite dimensional vector space, then left S -module $\text{Hom}^{-C}(X, A)$ is a finitely generated projective module, hence it can be truly interpreted as a module of sections on a non-commutative vector bundle in the sense of Connes (cf. [12]). In particular, in this way one can study some aspects of the K -theory of principal extensions (Chern-Galois characters). From the non-commutative geometry point of view it is therefore extremely interesting to study what additional properties must be imposed on a principal C -comodule M to make $\text{Hom}^{-C}(M, W)$ a finitely generated left S -module for any right C -comodule W that is finitely generated and projective as an A -module. One of the possibilities is explored in the following

Proposition 6.6. *Let k be a field, M be a Galois right C -comodule with the endomorphism ring $S = \text{End}^{-C}(M)$. View $S \otimes_k M$ as a right C -comodule via $S \otimes_k \varrho^M$ and $M^* \otimes_k S$ as a left C -comodule via ${}^{M^*} \varrho \otimes_k S$.*

- (1) *If ${}_S M$ is faithfully flat and there exists a right S -module left C -comodule section of the action $M^* \otimes_k S \rightarrow M^*$, then, for every left C -comodule V that is finitely generated projective as a left A -module, $\text{Hom}^{C-}(V, M^*)$ is a finitely generated projective right S -module.*
- (2) *If M_S^* is faithfully flat and there exists a left S -module right C -comodule section of the action $S \otimes_k M \rightarrow M$, then, for every right C -comodule W that is finitely generated projective as a right A -module, $\text{Hom}^{-C}(W, M)$ is a finitely generated projective left S -module.*

Proof. (1) If a left C -comodule V is a finitely generated projective left A -module then *V is a right C -comodule and $\text{Hom}^{C-}(V, M^*) \simeq {}^*V \square_C M^*$ as right S -modules. Thus one can consider the following chain of isomorphisms of right A -modules

$$\text{Hom}^{C-}(V, M^*) \otimes_S M \simeq ({}^*V \square_C M^*) \otimes_S M \simeq {}^*V \square_C (M^* \otimes_S M) \simeq {}^*V \square_C \mathcal{C} \simeq {}^*V.$$

The second isomorphism is the consequence of the fact that ${}_S M$ is flat, the third one follows from the bijectivity of the canonical map can_M . Since V is a finitely generated left A -module, its dual is a finitely generated right A -module. Thus $\text{Hom}^{\mathcal{C}^-}(V, M^*) \otimes_S M$ is a finitely generated right A -module. Therefore, there exists a finite number of elements $x_{ij} \in \text{Hom}^{\mathcal{C}^-}(V, M^*)$ such that $\text{Hom}^{\mathcal{C}^-}(V, M^*) \otimes_S M$ is generated by $\sum_j x_{ij} \otimes e^j$, where e^j are generators of M_A . Let X be a submodule of $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ generated by the x_{ij} . Applying $- \otimes_S M$ to the inclusion $0 \rightarrow X \rightarrow \text{Hom}^{\mathcal{C}^-}(V, M^*)$ one obtains a right A -module surjection $X \otimes_S M \rightarrow \text{Hom}^{\mathcal{C}^-}(V, M^*) \otimes_S M \rightarrow 0$. Since ${}_S M$ is faithfully flat, also the inclusion $X \rightarrow \text{Hom}^{\mathcal{C}^-}(V, M^*)$ is a surjection, and since X is a finitely generated right S -module, so is $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ (cf. proof of [3, Ch. 1§3 Prop. 11]).

Let $\sigma : M^* \rightarrow M^* \otimes_k S$ be a right S -linear left \mathcal{C} -colinear section of the action $M^* \otimes_k S \rightarrow M^*$. Write $\theta : \text{Hom}^{\mathcal{C}^-}(V, M^*) \rightarrow {}^*V \square_{\mathcal{C}} M^*$ for the isomorphism of right S -modules $\text{Hom}^{\mathcal{C}^-}(V, M^*) \simeq {}^*V \square_{\mathcal{C}} M^*$. Then $\sigma_V = (\theta^{-1} \otimes_k S) \circ ({}^*V \square_{\mathcal{C}} \sigma) \circ \theta$ is a right S -module section of the multiplication map $\text{Hom}^{\mathcal{C}^-}(V, M^*) \otimes_k S \rightarrow \text{Hom}^{\mathcal{C}^-}(V, M^*)$. Thus $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ is a projective module. This completes the proof of the assertion that $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ is a finitely generated projective right S -module.

(2) Follows by similar arguments as part (1). \square

Unfortunately, the conditions in Proposition 6.6 are too strong to cover the case of principal extensions. Thus the problem of finding suitable conditions remains open. Another possible criterion is given in Proposition 7.4.

Theorem 6.1 implies also the following reflexivity property of Galois comodules.

Corollary 6.7. *Let M and N be Galois right \mathcal{C} -comodules and set $S = \text{End}^{-\mathcal{C}}(M)$ and $T = \text{End}^{-\mathcal{C}}(N)$. If ${}_S M$ and ${}_T N$ are flat, then*

$$\text{Hom}^{-\mathcal{C}}(M, N) \simeq \text{Hom}_{-T}(\text{Hom}_{-S}(\text{Hom}^{-\mathcal{C}}(M, N), S), T) = ((\text{Hom}^{-\mathcal{C}}(M, N))^*)^*,$$

as (T, S) -bimodules.

Proof. Apply Theorem 6.1 twice, and notice that the isomorphism in Theorem 6.1,

$$\text{Hom}^{-\mathcal{C}}(N, M) \rightarrow \text{Hom}_{-S}(\text{Hom}^{-\mathcal{C}}(M, N), S), \quad f \mapsto [\phi \mapsto f \circ \phi],$$

is an (S, T) -bimodule map. \square

7. RELATIVELY INJECTIVE GALOIS COMODULES

For any right \mathcal{C} -comodule M , the comodule endomorphism ring $S = \text{End}^{-\mathcal{C}}(M)$ is a subring of the module endomorphism ring $\hat{S} = \text{End}_{-A}(M)$. Thus there is a ring

extension $S \rightarrow \hat{S}$. The properties of this extension capture properties of the comodule M . This is most profound in the case of Galois comodules.

Recall that a right \mathcal{C} -comodule M is called a (\mathcal{C}, A) -*injective comodule* or an A -*relatively injective comodule* if, for every \mathcal{C} -comodule map $i : N \rightarrow L$ that has a retraction in \mathbf{M}_A , every diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & L \\ & \searrow f & \\ & & M \end{array}$$

in $\mathbf{M}^{\mathcal{C}}$ can be completed commutatively by some $g : L \rightarrow M$ in $\mathbf{M}^{\mathcal{C}}$. Equivalently, M is a (\mathcal{C}, A) -injective comodule if the coaction $\varrho^M : M \rightarrow M \otimes_A \mathcal{C}$ has a right \mathcal{C} -comodule retraction $\pi_M : M \otimes_A \mathcal{C} \rightarrow M$. Here $M \otimes_A \mathcal{C}$ is a right \mathcal{C} -comodule with the coaction $M \otimes_A \Delta_{\mathcal{C}}$. (\mathcal{C}, A) -injective left comodules are defined in a symmetric way.

Recall also that a ring extension $S \rightarrow R$ is called a *split extension* provided there exists an (S, S) -bimodule map $\sigma : R \rightarrow S$ such that $\sigma(1_R) = 1_S$. We begin with the following simple

Lemma 7.1.

(1) *Let M be a Galois right \mathcal{C} -comodule and let $S = \text{End}^{-\mathcal{C}}(M)$. Then*

$$\text{Hom}^{-\mathcal{C}}(\mathcal{C}, M) \simeq \text{Hom}_{-S}(M^*, S),$$

as left S -modules.

(2) *Let N be a Galois left \mathcal{C} -comodule and let $T = \text{End}^{\mathcal{C}-}(N)$. Then*

$$\text{Hom}^{\mathcal{C}-}(\mathcal{C}, N) \simeq \text{Hom}_{T-}(N^*, T),$$

as right T -modules.

In particular, for a Galois right \mathcal{C} -comodule M , $\text{Hom}^{\mathcal{C}-}(\mathcal{C}, M^) \simeq \text{Hom}_{S-}(M, S)$.*

Proof. We only prove (1) as (2) can be obtained by the left-right symmetry. This is a consequence of the following chain of isomorphisms:

$$\begin{aligned} \text{Hom}^{-\mathcal{C}}(\mathcal{C}, M) &\simeq \text{Hom}^{-\mathcal{C}}(M^* \otimes_S M, M) \\ &\simeq \text{Hom}_{-S}(M^*, \text{Hom}^{-\mathcal{C}}(M, M)) \simeq \text{Hom}_{-S}(M^*, S). \end{aligned}$$

The first isomorphism is obtained by applying $\text{Hom}^{-\mathcal{C}}(-, M)$ to the canonical map can_M , while the second is the hom-tensor relation for modules and comodules (cf. [9, 18.10(2)]). The final assertion follows from the observation that M^* is a Galois left \mathcal{C} -comodule and from assertion (2). \square

Theorem 7.2.

- (1) Let M be a Galois right \mathcal{C} -comodule, let $S = \text{End}^{-\mathcal{C}}(M)$ and $\hat{S} = \text{End}_{-A}(M)$. Then M is a (\mathcal{C}, A) -injective comodule if and only if there exists a right S -module map $\sigma : \hat{S} \rightarrow S$ such that $\sigma(1_{\hat{S}}) = 1_S$.
- (2) Let N be a Galois left \mathcal{C} -comodule, let $T = \text{End}^{\mathcal{C}}(N)$ and $\hat{T} = \text{End}_{A-}(N)$. Then N is a (\mathcal{C}, A) -injective comodule if and only if there exists a left T -module map $\tau : \hat{T} \rightarrow T$ such that $\tau(1_{\hat{T}}) = 1_T$.
- (3) Let M be a Galois right \mathcal{C} -comodule, let $S = \text{End}^{-\mathcal{C}}(M)$ and $\hat{S} = \text{End}_{-A}(M)$. The following statements are equivalent
- (a) $S \rightarrow \hat{S}$ is a split extension;
 - (b) the right coaction ϱ^M has a right \mathcal{C} -comodule left S -module retraction;
 - (c) the left coaction ${}^M\varrho$ has a left \mathcal{C} -comodule right S -module retraction.
- In particular, if $S \rightarrow \hat{S}$ is a split extension, then M is a (\mathcal{C}, A) -injective right \mathcal{C} -comodule and M^* is a (\mathcal{C}, A) -injective left \mathcal{C} -comodule.

Proof. (1) Consider the following chain of isomorphisms

$$\begin{aligned} \text{Hom}_{-S}(\hat{S}, S) &\simeq \text{Hom}_{-S}(M \otimes_A M^*, S) \simeq \text{Hom}_{-A}(M, \text{Hom}_{-S}(M^*, S)) \\ &\simeq \text{Hom}_{-A}(M, \text{Hom}^{-\mathcal{C}}(\mathcal{C}, M)) \simeq \text{Hom}^{-\mathcal{C}}(M \otimes_A \mathcal{C}, M). \end{aligned}$$

The first isomorphism is the consequence of the canonical isomorphism $\text{End}_{-A}(M) \simeq M \otimes_A M^*$ that holds for any finitely generated projective module. The second and the fourth isomorphisms are the hom-tensor relations for modules and comodules respectively. The third isomorphism is obtained by applying $\text{Hom}_{-A}(M, -)$ to the isomorphism in Lemma 7.1. Explicitly, the composite isomorphism $\Theta : \text{Hom}_{-S}(\hat{S}, S) \rightarrow \text{Hom}^{-\mathcal{C}}(M \otimes_A \mathcal{C}, M)$ comes out as follows. First, for any $c \in \mathcal{C}$ write

$$\text{can}_M^{-1}(c) = \sum c^{[1]} \otimes c^{[2]} \in M^* \otimes_S M.$$

Then, for all $\sigma \in \text{Hom}_{-S}(\hat{S}, S)$, $m \in M$, $c \in \mathcal{C}$, $\pi \in \text{Hom}^{-\mathcal{C}}(M \otimes_A \mathcal{C}, M)$ and $\hat{s} \in \hat{S}$, the isomorphism Θ and its inverse Θ^{-1} read

$$\Theta(\sigma)(m \otimes c) = \sum \sigma(m c^{[1]}(-))(c^{[2]}), \quad \Theta^{-1}(\pi)(\hat{s}) = \pi \circ (\hat{s} \otimes_A \mathcal{C}) \circ \varrho^M.$$

The fact that Θ and Θ^{-1} are mutual inverses can also be shown directly as follows. First note that $\text{can}_M^{-1}(c)$ has the following properties:

$$\sum c^{[1]}(c^{[2]}_{(0)})c^{[2]}_{(1)} = c, \quad \forall c \in \mathcal{C}, \quad (A)$$

for $(\text{can}_M \circ \text{can}_M^{-1})(c) = c$, and

$$\xi \otimes_S m = \sum \xi(m_{(0)})m_{(1)}^{[1]} \otimes_S m_{(1)}^{[2]}, \quad \forall m \in M, \xi \in M^*, \quad (B)$$

for $(\text{can}_M^{-1} \circ \text{can}_M)(\xi \otimes m) = \xi \otimes m$. Let $\{e^i \in M, \xi^i \in M^*\}_{i=1, \dots, n}$ be the dual basis of M_A . In view of the isomorphism $\hat{S} \simeq M \otimes_A M^*$, $\hat{s} \mapsto \sum_i \hat{s}(e^i) \otimes \xi^i$, $m \otimes \xi \mapsto m\xi(-)$, the property (B) implies that, for all $\hat{s} \in \hat{S}$ and $m \in M$,

$$\hat{s} \otimes_S m = \sum \hat{s}(m_{(0)})m_{(1)}^{[1]}(-) \otimes_S m_{(1)}^{[2]}. \quad (C)$$

With these equalities at hand we can compute, for all $\sigma \in \text{Hom}_{-S}(\hat{S}, S)$, $\hat{s} \in \hat{S}$ and $m \in M$,

$$\begin{aligned} (\Theta^{-1} \circ \Theta)(\sigma)(\hat{s})(m) &= \Theta(\sigma)\left(\sum \hat{s}(m_{(0)}) \otimes m_{(1)}\right) \\ &= \sum \sigma(\hat{s}(m_{(0)})m_{(1)}^{[1]}(-))(m_{(1)}^{[2]}) = \sigma(\hat{s})(m), \end{aligned}$$

by the property (C). On the other hand the use of property (A) entails that, for all $\pi \in \text{Hom}^{-C}(M \otimes_A \mathcal{C}, M)$, $m \in M$ and $c \in \mathcal{C}$,

$$\begin{aligned} (\Theta \circ \Theta^{-1})(\pi)(m \otimes c) &= \sum \Theta^{-1}(\pi)(mc^{[1]}(-))(c^{[2]}) \\ &= \sum \pi(m \otimes c^{[1]}(c^{[2]}_{(0)}))c^{[2]}_{(1)} = \pi(m \otimes c). \end{aligned}$$

Thus Θ and Θ^{-1} are mutual inverses as claimed on the basis of the chain of isomorphisms displayed at the beginning of the proof.

Suppose that $\sigma \in \text{Hom}_{-S}(\hat{S}, S)$ has the property $\sigma(1_{\hat{S}}) = 1_S$. Then, for all $m \in M$,

$$\Theta(\sigma)\left(\sum m_{(0)} \otimes m_{(1)}\right) = \sum \sigma(m_{(0)}m_{(1)}^{[1]}(-))(m_{(1)}^{[2]}) = \sigma(1_{\hat{S}})(m) = m,$$

where the second equality follows from property (C), by setting $\hat{s} = 1_{\hat{S}}$. Thus $\Theta(\sigma)$ is a retraction of ϱ^M , and hence M is a (\mathcal{C}, A) -injective comodule. Conversely, if M is a (\mathcal{C}, A) -injective comodule and π_M is a retraction of ϱ^M , then, for all $m \in M$,

$$\Theta^{-1}(\pi_M)(1_{\hat{S}})(m) = \pi_M\left(\sum m_{(0)} \otimes m_{(1)}\right) = m,$$

i.e., $\Theta^{-1}(\pi_M)(1_{\hat{S}}) = 1_S$, as required.

The assertion (2) is proven in the similar way to the proof of (1).

(3) The equivalence (a) \Leftrightarrow (b) follows from the observation that the maps Θ and Θ^{-1} , constructed in the proof of assertion (1), preserve the left S -linearity. Indeed, if $\sigma \in \text{Hom}_{-S}(\hat{S}, S)$ is left S -linear then for all $s \in S$, $m \in M$ and $c \in \mathcal{C}$,

$$\begin{aligned} \Theta(\sigma)(s(m) \otimes c) &= \sum \sigma(s(m)c^{[1]}(-))(c^{[2]}) = \sum \sigma(s(mc^{[1]}(-)))(c^{[2]}) \\ &= \sum s\sigma(mc^{[1]}(-))(c^{[2]}) = s\Theta(\sigma)(m \otimes c), \end{aligned}$$

so that $\Theta(\sigma)$ is also left S -linear. On the other hand, $\pi \in \text{Hom}^{-\mathcal{C}}(M \otimes_A \mathcal{C}, M)$ is left S -linear provided for all $s \in S$, $\pi \circ (s \otimes_A \mathcal{C}) = s \circ \pi$. If this is so, then the definition of $\Theta^{-1}(\pi)$ immediately implies that $\Theta^{-1}(\pi)$ is also left S -linear. In view of this and part (1), the coaction has a right \mathcal{C} -colinear left S -linear retraction if and only if there exists $\sigma \in \text{Hom}_{S,S}(\hat{S}, S)$ such that $\sigma(1_{\hat{S}}) = 1_S$, i.e., if and only if $S \rightarrow \hat{S}$ is a split extension.

The equivalence (a) \Leftrightarrow (c) is proven in a similar manner by noting that M^* is a Galois left \mathcal{C} -comodule, and $\text{End}_{A-}(M^*) \simeq \text{End}_{-A}(M)$ and $\text{End}^{\mathcal{C}-}(M^*) \simeq \text{End}^{-\mathcal{C}}(M)$. The final assertion is obvious. \square

Theorem 7.2 leads to the following conditions for a Galois right comodule M to be faithfully flat as a left S -module.

Proposition 7.3. *Let M be a Galois right \mathcal{C} -comodule and let $S = \text{End}^{-\mathcal{C}}(M)$ and $\hat{S} = \text{End}_{-A}(M)$. If either*

- (a) *${}_S M$ is flat and there exists $\sigma \in \text{Hom}_{S-}(\hat{S}, S)$ such that $\sigma(1_{\hat{S}}) = 1_S$, or*
- (b) *${}_A \mathcal{C}$ is flat and $S \rightarrow \hat{S}$ is a split extension,*

then M is a faithfully flat left S -module.

Proof. (a) If ${}_S M$ is flat, then ${}_A \mathcal{C}$ is flat and the counit of adjunction (the evaluation map) $\varphi_N : \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M \rightarrow N$ is bijective for any right \mathcal{C} -comodule N by Theorem 2.1(1). Since M is a Galois right \mathcal{C} -comodule, the dual module M^* is a Galois left \mathcal{C} -comodule. Furthermore, $\text{End}_{A-}(M^*) \simeq \text{End}_{-A}(M)$ and $\text{End}^{\mathcal{C}-}(M^*) \simeq \text{End}^{-\mathcal{C}}(M)$ via the isomorphism Γ_M described in equation (4.3), and thus the existence of σ implies that M^* is a (\mathcal{C}, A) -injective comodule by Theorem 7.2(2). Let $\pi_{M^*} \in \text{Hom}^{\mathcal{C}-}(\mathcal{C} \otimes_A M^*, M^*)$ be a retraction of the left coaction ${}^{M^*}\varrho$ corresponding to σ as in Theorem 7.2. Taking into account Γ_M (so that the elements of S and \hat{S} are understood as maps on M) and the relationship between \mathcal{C} -coactions on M and M^* described in equation (4.1), this correspondence has the following explicit form, for all $\hat{s} \in \hat{S}$,

$$\sigma(\hat{s}) = \sum_i \hat{s}(e^i)_{(0)} \pi_{M^*}(\hat{s}(e^i)_{(1)} \otimes \xi^i)(-), \quad (*)$$

where $e^i \in M$, $\xi^i \in M^*$ is the finite dual basis of M . Let, for all $X \in \mathbf{M}_S$,

$$\nu_X : X \rightarrow \text{Hom}^{-\mathcal{C}}(M, X \otimes_S M), \quad x \mapsto [m \mapsto x \otimes m]$$

denote the unit of the adjunction. For any $f \in \text{Hom}^{-\mathcal{C}}(M, X \otimes_S M)$ and $m \in M$, write $f(m) = \sum f(m)^{(1)} \otimes f(m)^{(2)} \in X \otimes_S M$. We claim that the map

$$\nu_X^{-1} : \text{Hom}^{-\mathcal{C}}(M, X \otimes_S M) \rightarrow X, \quad f \mapsto \sum_i f(e^i)^{(1)} \sigma(f(e^i)^{(2)} \xi^i(-)),$$

is the inverse of ν_X . Indeed, for all $x \in X$,

$$\begin{aligned} (\nu_X^{-1} \circ \nu_X)(x) &= \sum_i \nu_X(x) (e^i)^{(1)} \sigma(\nu_X(x) (e^i)^{(2)} \xi^i(-)) \\ &= \sum_i x \sigma(e^i \xi^i(-)) = x \sigma(1_{\hat{S}}) = x. \end{aligned}$$

On the other hand, in the view of the correspondence $(*)$ and the fact that f is a right \mathcal{C} -comodule map,

$$\begin{aligned} \nu_X^{-1}(f) &= \sum_i f(e^i)^{(1)} (f(e^i)^{(2)})_{(0)} \pi_{M^*}(f(e^i)^{(2)})_{(1)} \otimes \xi^i(-)) \\ &= \sum_i f(e^i_{(0)})^{(1)} (f(e^i_{(0)})^{(2)}) \pi_{M^*}(e^i_{(1)} \otimes \xi^i(-)). \end{aligned}$$

Therefore, for all $m \in M$, $f \in \text{Hom}^{-\mathcal{C}}(M, X \otimes_S M)$,

$$\begin{aligned} (\nu_X \circ \nu_X^{-1})(f)(m) &= \sum_i f(e^i_{(0)})^{(1)} (f(e^i_{(0)})^{(2)}) \pi_{M^*}(e^i_{(1)} \otimes \xi^i(-)) \otimes_S m \\ &= \sum_i f(e^i_{(0)})^{(1)} \otimes_S f(e^i_{(0)})^{(2)} \pi_{M^*}(e^i_{(1)} \otimes \xi^i)(m) \\ &= \sum_i f(e^i)^{(1)} \otimes_S f(e^i)^{(2)} \pi_{M^*}(\xi^i_{(-1)} \otimes \xi^i_{(0)})(m) \quad (\text{by eq. (4.2)}) \\ &= \sum_i f(e^i) \xi^i(m) = f(m), \end{aligned}$$

where the penultimate equality follows from the fact that π_{M^*} is a retraction of the coaction of M^* .

In this way we have proven that $\text{Hom}^{-\mathcal{C}}(M, -) : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_S$ is an equivalence. We have already observed that \mathcal{C} is a flat left A -module. Now Theorem 2.1(2) implies that M is a faithfully flat left S -module.

(b) Since $S \rightarrow \hat{S}$ is a split extension, M^* is a (\mathcal{C}, A) -injective left \mathcal{C} -comodule by Theorem 7.2. Furthermore, the retraction of the left coaction ${}^{M^*}\varrho$ is a right S -module map. Hence, for all $N \in \mathbf{M}^{\mathcal{C}}$, there is an isomorphism (the tensor-cotensor relation)

$$N \square_{\mathcal{C}}(M^* \otimes_S M) \simeq (N \square_{\mathcal{C}} M^*) \otimes_S M,$$

(cf. [9, 21.4, 21.5]). Furthermore, M_A is finitely generated projective, so $N \square_{\mathcal{C}} M^* \simeq \text{Hom}^{-\mathcal{C}}(M, N)$, and therefore $N \square_{\mathcal{C}}(M^* \otimes_S M) \simeq \text{Hom}^{-\mathcal{C}}(M, N) \otimes_S M$. Thus one can

consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}^{-\mathcal{C}}(M, N) \otimes_S M & \longrightarrow & N \otimes_A M^* \otimes_S M & \longrightarrow & N \otimes_A \mathcal{C} \otimes_A M^* \otimes_S M \\
& & \downarrow \varphi_N & & \downarrow N \otimes_A \mathrm{can}_M & & \downarrow N \otimes_A \mathcal{C} \otimes_A \mathrm{can}_M \\
0 & \longrightarrow & N & \xrightarrow{\varrho^N} & N \otimes_A \mathcal{C} & \longrightarrow & N \otimes_A \mathcal{C} \otimes_A \mathcal{C}
\end{array}$$

The top row is the defining sequence of $N \square_{\mathcal{C}}(M^* \otimes_S M) \simeq \mathrm{Hom}^{-\mathcal{C}}(M, N) \otimes_S M$, hence it is an exact sequence. The last map in the bottom row is $\varrho^N \otimes_A \mathcal{C} - N \otimes_A \Delta_{\mathcal{C}}$, hence the sequence is exact by the coassociativity of the coaction. Since can_M is an isomorphism, so are $N \otimes_A \mathrm{can}_M$ and $N \otimes_A \mathcal{C} \otimes_A \mathrm{can}_M$. Thus φ_N is an isomorphism for any right \mathcal{C} -comodule N . By the same arguments as in part (a) we conclude that $\mathrm{Hom}^{-\mathcal{C}}(M, -)$ is an equivalence. Since ${}_A \mathcal{C}$ is flat, Theorem 2.1(2) implies that M is a faithfully flat left S -module. \square

A special case of Proposition 7.3 is obtained by taking A a coalgebra-Galois \mathcal{C} -extension, $\mathcal{C} = A \otimes_k C$ and $M = A$. In this case one derives [4, Proposition 4.4]. Furthermore, if A is a Hopf-Galois H -extension, $\mathcal{C} = A \otimes_k H$ and $M = A$, Proposition 7.3 gives [13, Theorem 2.11].

Combined with Theorem 7.2, the criterion of faithful flatness in Proposition 7.3(b) leads also to the following criterion for modules associated to a Galois comodule to be finitely generated projective modules.

Proposition 7.4. *Let k be a field, and let M be a Galois right \mathcal{C} -comodule, $S = \mathrm{End}^{-\mathcal{C}}(M)$ and $\hat{S} = \mathrm{End}_{-A}(M)$. Suppose that $S \rightarrow \hat{S}$ is a split extension. Then:*

- (1) *If M_S^* is projective (i.e., M^* is a principal left comodule) and ${}_A \mathcal{C}$ is flat, then, for any left \mathcal{C} -comodule V that is finitely generated projective as a left A -module, $\mathrm{Hom}^{\mathcal{C}^-}(V, M^*)$ is a finitely generated projective right S -module.*
- (2) *If ${}_S M$ is projective (i.e., M is a principal right comodule) and \mathcal{C}_A is flat, then, for any right \mathcal{C} -comodule W that is finitely generated projective as a right A -module, $\mathrm{Hom}^{-\mathcal{C}}(W, M)$ is a finitely generated projective left S -module.*

Proof. (1) By Proposition 7.3, M is a faithfully flat left S -module. Hence by the same means as in the proof of Proposition 6.6 one proves that $\mathrm{Hom}^{\mathcal{C}^-}(V, M^*)$ is a finitely generated right S -module.

Let $\sigma : M^* \rightarrow M^* \otimes_k S$ be a right S -module section of the multiplication map. For all $\xi \in M^*$, write $\sigma(\xi) = \sum \xi^{(1)} \otimes \xi^{(2)} \in M^* \otimes_k S$, so that, for all $m \in M$, $\sum \xi^{(1)}(\xi^{(2)}(m)) = \xi(m)$. Let $e_i \in V$, $\eta_i \in {}^*V$ be a dual basis of V . Define a right

S -linear map $\sigma_V : \text{Hom}_{A-}(V, M^*) \rightarrow \text{Hom}_{A-}(V, M^*) \otimes_k S$, by

$$f \mapsto \sum_i \eta_i(-) \sigma(f(e_i)) = \sum_i \eta_i(-) f(e_i)^{(1)} \otimes f(e_i)^{(2)}.$$

Then, for all $v \in V$ and $m \in M$,

$$\begin{aligned} \sum_i ((\eta_i(-) f(e_i)^{(1)}) \cdot f(e_i)^{(2)})(m)(v) &= \sum_i \eta_i(v) f(e_i)^{(1)}(f(e_i)^{(2)}(m)) \\ &= \sum_i \eta_i(v) f(e_i)(m) \\ &= \sum_i f(\eta_i(v) e_i)(m) = f(v)(m). \end{aligned}$$

This means that σ_V is a section of the product $\text{Hom}_{A-}(V, M^*) \otimes_k S \rightarrow \text{Hom}_{A-}(V, M^*)$, and hence $\text{Hom}_{A-}(V, M^*)$ is a projective right S -module.

By Theorem 7.2(3) there is a right S -linear, left \mathcal{C} -colinear retraction $\pi : \mathcal{C} \otimes_A {}^*M \rightarrow {}^*M$ of the coaction ${}^{M^*}\varrho$. We claim that the map

$$\pi_V : \text{Hom}_{A-}(V, M^*) \rightarrow \text{Hom}^{\mathcal{C}^-}(V, M^*), \quad f \mapsto \pi \circ (\mathcal{C} \otimes_A f) \circ {}^V\varrho,$$

is a right S -linear retraction of the defining inclusion $\text{Hom}^{\mathcal{C}^-}(V, M^*) \subseteq \text{Hom}_{A-}(V, M^*)$. Indeed, $\pi_V(f)$ is left \mathcal{C} -colinear, since it is a composition of left \mathcal{C} -colinear maps. Furthermore, for all $f \in \text{Hom}_{A-}(V, M^*)$, $s \in S$ and $v \in V$,

$$\begin{aligned} \pi_V(fs)(v) &= \sum \pi(v_{(-1)} \otimes (fs)(v_{(0)})) = \sum \pi(v_{(-1)} \otimes f(v_{(0)}) \circ s) \\ &= \sum \pi(v_{(-1)} \otimes f(v_{(0)})) \circ s = (\pi_V(f)s)(v), \end{aligned}$$

where the penultimate equality is a consequence of the right S -linearity of π . Thus π_V is a right S -linear map. Finally, if $f \in \text{Hom}^{\mathcal{C}^-}(V, M^*)$, then

$$\pi \circ (\mathcal{C} \otimes_A f) \circ {}^V\varrho = \pi \circ {}^{M^*}\varrho \circ f = f,$$

since π is a section of the coaction. Therefore, $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ is a direct summand of a projective right S -module, hence a projective module. This completes the proof that $\text{Hom}^{\mathcal{C}^-}(V, M^*)$ is a finitely generated projective right S -module, as required.

(2) This is proven in the analogous way to the proof of part (1). \square

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